The Twistor Equation, Moment Forms and Compatible Complex Structures on a Quaternion-Kähler Manifold

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Abstract

Let (M^{4n}, g, Q) be a quaternion-Kähler manifold with non-zero scalar curvature K. Extending previous results of Salamon, Galicki and Lawson we describe a correspondence among conformally balanced complex structures J which are compatible with the quaternionic structure Q, Killing vector fields X, self-dual 2-forms μ satisfying the twistor equation and eigenforms of the Laplacian acting on 2-forms.

1 Introduction

We report on some recent work in collaboration with D. V. Alekseevsky and M. Pontecorvo ([AMP1], [AMP2], [AMP3]). The first motivation of our research was to investigate the existence of almost complex structures which are defined on an *almost quaternionic manifold* (M^{4n}, Q) and which are compatible with the almost quaternionic structure Q. General results was obtained in [AMP1] where we defined the twistor fibration $t : Z \longrightarrow M^{4n}$ in this wider context by generalizing Salamon construction made for a *quaternionic* manifold.

Then we passed to consider the stronger notion of a quaternion-Kähler manifold (M^{4n}, g, Q) and by going deeper in the investigation we discovered links between compatible complex structures and several interesting objects which are related to the geometry of (M^{4n}, g, Q) , as shortly mentioned in the title, see [AMP2] and [AMP3].

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2 A quaternion-Kähler manifold (M^{4n}, g, Q)

Let (M^{4n}, g, Q) denote a *quaternion-Kähler manifold*, with $n \geq 2$. Q is the 3-dimensional subbundle of the bundle of endomorphisms which gives the quaternionic structure: it is locally spanned by an *admissible basis* $H = (J_1, J_2, J_3)$, that is a triple of fields of endomorphisms such that

$$J_\alpha^2 = -Id \qquad , \qquad J_\alpha J_\beta = -J_\beta J_\alpha = J_\gamma \ ,$$

where (α, β, γ) is a circular permutation of $(1, 2, 3)$.

The Riemannian metric g is a *Q-Hermitian metric*, that is

$$g(JX, Y) + g(X, JY) = 0 \qquad \forall J \in Q$$

for any vector fields X, Y on M^{4n}. The Levi-Civita connection $\nabla = \nabla^g$ of g preserves Q.

Let recall also that g is Einstein, that is

$$Ric(g) = (K/4n)g \ ,$$

where K is the scalar curvature. We denote $\nu = K/4n(n+2)$ the *reduced scalar curvature.*

One defines the bundle $\Lambda^2_+ \longrightarrow M^{4n}$ of *self-dual 2-forms* associated to Q by $\Lambda^2_+|_x = (g \circ Q)_x$, $x \in M^{4n}$. If $H = (J_\alpha)$ is an admissible basis of Q then the three forms

$$F_\alpha = g \circ J_\alpha = g(J_\alpha \cdot, \cdot) \qquad (\alpha = 1, 2, 3)$$

give a (local) frame (F_1, F_2, F_3) for Λ^2_+ and

$$\nabla F_\alpha = \omega_\gamma \otimes F_\beta - \omega_\beta \otimes F_\gamma \ ,$$

where the ω_α, $\alpha = 1, 2, 3$, are 1-forms.

We recall that the following *fundamental identities* hold:

$$d\omega_\alpha + \omega_\beta \wedge \omega_\gamma = -\nu F_\alpha \ , \qquad (\alpha = 1, 2, 3)$$

and moreover

$$\nu F_\alpha(X, Y) = \frac{1}{2n} Tr[J_\alpha R(X, Y)] \ , \qquad (\alpha = 1, 2, 3)$$

where R is the curvature tensor of ∇ (see [AM2]).

The global 4-form Ω defined by

$$\Omega = \sum_\alpha F_\alpha \wedge F_\alpha$$

is the *fundamental 4-form.*

If there exists a *global* admissible basis $H = (J_\alpha)$ which is parallel, that is the J_α, $\alpha = 1, 2, 3$, are globally defined parallel complex structures, then $\nu = 0$ and (M^{4n}, g, H) is an *HyperKähler manifold.*

3 Compatible (almost) complex structures

Let (M^{4n}, g, Q) be a quaternion-Kähler manifold.

A section $J : U \longrightarrow Q$, $J^2 = -Id$ is called a **compatible almost complex structure (compatible a.c.s.)** defined on $U \subset M^{4n}$.

Several natural questions arise concerning the existence on (M^{4n}, g, Q) of compatible almost complex structure J which satisfy some special property. They were discussed in [AMP2] and here we summarize the main results.

A first basic result is the following one.

Theorem. *Assume (M^{4n}, g, Q) is a simply connected complete irreducible quaternion-Kähler manifold with reduced scalar curvature ν. Let J be a parallel (not necessarily compatible) complex structure on M^{4n}. Then*

(1) *J is compatible if and only if $\nu = 0$, and $J = \sum_{\alpha=1}^{3} a_\alpha J_\alpha$ where $a_\alpha =$ const, $\sum a_\alpha^2 = 1$ and $H = (J_\alpha)$ is a HyperKähler structure on M^{4n},*

(2) *otherwise (M^{4n}, g) is isometric to the quaternionic symmetric space $G_2(\mathbb{C}^m)$ or to its dual non compact symmetric space, and J is the unique parallel complex structure.*

Remark. A quaternion-Kähler manifold with $\nu = 0$ is a *locally HyperKähler* manifold.

If J is a compatible a.c.s. on (M^{4n}, g, Q) let $F = g \circ J$ be the *Kähler form* and $\theta = -\delta F \circ J$ the *Lee form* of J.

The following proposition summarizes local results.

Proposition. *Let J be a compatible almost complex structure on (M^{4n}, g, Q) . Then:*

(1) *if J is locally conformally symplectic (that is $dF = \alpha \wedge F$ for some 1-form α) then it is parallel (and $\nu = 0$).*

(2) *if J is cosymplectic (that is $\theta = 0$) then $\nu \leq 0$ and $\nu = 0$ if and only if J is parallel.*

(3) *if J is integrable then the differential $d\theta$ of the Lee form is Q-Hermitian and harmonic.*

(4) *any closed self-dual 2-form $\omega \in \Gamma(\Lambda^2_+)$ must be parallel.*

Among global results we have the following theorem.

Theorem. *Let (M^{4n}, g, Q) be complete and $\nu > 0$. Then no compatible almost complex structure J exists on M^{4n}.*

Remarks.

(1) If $\nu < 0$ an almost complex compatible J, even complex, can exist on a complete (M^{4n}, Q, g). In fact, on the hyperbolic quaternionic space $\mathcal{H}P_n$ there exist even globally defined hypercomplex bases $H = (J_\alpha)$, that is admissible bases such that the almost complex structures J_α, $(\alpha = 1, 2, 3)$ are integrable, see [P].

(2) A quaternion-Kähler manifold M^{4n} locally admits many compatible almost complex structures which are integrable, see [S2, p.130] and [AMP1]. Any such structure J defines a local section $J : \mathcal{U} \subseteq M^{4n} \longrightarrow Z$ of the twistor fibration $t : Z \longrightarrow M^{4n}$ by setting $\mathcal{U} \ni x \mapsto J_x \in Z$; furthermore the integrability of J corresponds to the fact that the image $W = J(\mathcal{U})$ is a complex submanifold of Z. Conversely, any complex submanifold $W \subset Z$ such that the projection $t : W \longrightarrow M$ is a diffeomorphism defines a compatible complex structure on $\mathcal{U} = t(W) \subseteq M^{4n}$.

(3) Pontecorvo proved that on a compact quaternion-Kähler manifold a compatible *complex* structure J is necessarily parallel, see [P], [AMP2].

A special interest concerns the existence on M^{4n}, even locally, of the following type of *complex* structures J.

Definition. *A Hermitian structure (g, J) is said to be* ***balanced*** *if (J is integrable and) the Lee form $\theta = 0$;* ***conformally balanced*** *if θ is exact.*

By referring to the fixed Riemannian metric g we also say for simplicity that J is balanced or, respectively, conformally balanced.

The interest on compatible conformally balanced complex structure on a quaternion-Kähler manifold (M^{4n}, g, Q) is related to the following results.

4. The twistor equation

Let V denote the tangent bundle TM. Then we have:

$$V^* \otimes \Lambda^2_+ = V^*_{(1)} \oplus (V^* \otimes \Lambda^2_+)_0 \ ,$$

where

$$V^*_{(1)}|_x = \{A^\xi = \sum_\alpha (\xi \circ J_\alpha) \otimes F_\alpha|_x \ , \ \xi \in V^*|_x\} \cong V^*|_x\} \qquad x \in M^{4n}$$

and $(V^* \otimes \Lambda^2_+)_0$ are the trace-free tensors.

Correspondingly to this decomposition we have the *Dirac operator* D,

$$D = p_1 \circ \nabla : \Gamma(\Lambda^2_+) \to \Gamma(V^*_{(1)})$$

and the *Twistor operator* $\bar{D}$,

$$\bar{D} = p_2 \circ \nabla : \Gamma(\Lambda^2_+) \to \Gamma((V^* \otimes \Lambda^2_+)_0) \ ,$$

where p_1 and p_2 are the projections onto the irreducible components of $V^* \otimes Q$. If a section σ of Λ^2_+ can be written $\sigma = fF$ for some smooth function f and F the Kähler form of a compatible almost complex structure J on M (and this is always possible on the open set $U = \{\sigma \neq 0\}$), then

$$D\sigma = -\frac{1}{3} \sum_\alpha \delta\sigma \circ J_\alpha \otimes J_\alpha \ ,$$

so *D can be identified with the codifferential $\delta : \Gamma(\Lambda^2_+) \longrightarrow \Gamma(\Lambda^1)$.*

Of course $\bar{D}\sigma = \nabla\sigma - D\sigma$.

Theorem. *A non-zero self-dual 2-form $\sigma = fJ$ satisfies the twistor equation $\bar{D}\sigma = 0$ if and only if J is conformally balanced with Lee form $\theta = d \log f^2$.*

Compatible complex conformally balanced structures J are also related to Killing vector fields.

Proposition. *Assume that a compatible complex structure J on (M^{4n}, g, Q) is conformally balanced with Lee form $\theta = d \log f^2$. Then $df \circ J \equiv \frac{1}{2} f\theta \circ J \equiv \frac{1}{2} f\delta F$ is a Killing 1-form - i.e. the dual vector field $X = (df \circ J)^\sharp$ is Killing. Furthermore, if $X = 0$ then the scalar curvature $\nu = 0$.*

Remark. S. Salamon previously related solutions of the twistor equation to Killing vector fields ([S1]).

5. The Hamiltonian 2-form of a Killing vector field

Let assume now that (M^{4n}, g, Q) is a quaternion-Kähler manifold with $\nu \neq 0$. Then one associates to any Killing vector field X on M^{4n} a self-dual 2-form μ^X, which is called the **Hamiltonian form** of X, defined as follows. Since at any point $x \in M^{4n}$ the covariant differential ∇X belongs to the Lie algebra $\mathfrak{hol}_x$ of the holonomy group ([Ko]) one has the decomposition

$$\nabla X = \sum_{\alpha=1}^{3} \rho_\alpha(X) J_\alpha + \bar{L}_X ,$$

where $H = (J_\alpha)$ is an admissible basis of Q,

$$\rho_\alpha(X) = -\frac{1}{4n} Tr(J_\alpha \nabla X) \qquad (\alpha = 1, 2, 3)$$

are locally defined functions and $\bar{L}_X \in sp(n) = \{A \mid A^t = -A\ ,\ [A, Q] = 0\}$.

Then one defines the Hamiltonian form of X as the self-dual form given by

$$\mu^X = \frac{2}{3\nu} \sum_{\alpha=1}^{3} \rho_\alpha(X) F_\alpha = \frac{2}{3\nu} pr_{\Lambda^2_+}(g \circ \nabla X) ,$$

where $pr_{\Lambda^2_+}$ is the orthogonal projection $\Lambda^2 \longrightarrow \Lambda^2_+$.

The map $X \mapsto \mu^X$ from Killing vector fields to self-dual forms is intimately connected with the "momentum map"

$$\mu : M^{4n} \longrightarrow \mathcal{K}^* \otimes Q ,$$

where $\mathcal{K}^*$ is the Lie algebra of Killing vector fields and

$$\langle \mu(m), X \rangle = \mu^X(m) \qquad \forall m \in M^{4n}, X \in \mathcal{K}$$

defined in [Ga], [GL]; see also [Sw].

The proposition below establishes the properties of the Hamiltonian forms and is a more explicit version of results of Salamon in [S1] and of [GL].

Proposition. [S1] [S2] [GL] [Sw]. *Let (M^{4n}, g, Q) be a quaternion-Kähler manifold of reduced scalar curvature $\nu \neq 0$.*

(1) *If X is a Killing vector field the Hamiltonian form μ^X of X satisfies the twistor equation $\bar{D}\mu^X = 0$ and the dual 1-form $g \circ X = \delta\mu^X$.*

(2) *If σ is a self-dual 2-form satisfying the twistor equation then the vector field $X = (\delta\sigma)^\sharp$ dual to the 1-form $\delta\sigma$ is a Killing vector field and σ is the Hamiltonian form of X, $\sigma = \mu^X$.*

(3) *If X is a Killing vector field then $d\mu^X = \frac{1}{3}(i_X \Omega)$, where Ω is the fundamental 4-form.*

(4) *If X is a Killing vector field then $\nabla\mu^X = \frac{1}{3}\sum_{\alpha=1}^{3}(i_X F_\alpha) \otimes F_\alpha$.*

Remark. The compatible complex structure on the open set $\{\mu^X \neq 0\}$ has been considered by Battaglia in [B].

6 Self-dual eigenforms of Laplacian

The following proposition, which was proved in [AMP3], relates Killing fields and compatible conformally balanced complex structures to eigenforms of the Laplacian acting on 2-forms.

Theorem [AMP3]**.** *Let (M^{4n}, g, Q) be a quaternion-Kähler manifold with $\nu \neq 0$. Let X be a Killing vector field. Then the associated Hamiltonian 2-form $\mu = \mu^X$ is an eigenform of the Laplacian with the eigenvalue $\lambda = 2(n+2)\nu$, i.e. $\lambda = \frac{\mathrm{K}}{2n}$.*

Proof. It is sufficient to prove that a self-dual 2-form $\sigma = fF$ where F is the Kähler form of a conformally balanced complex structure J with Lee form $\theta = dlog f^2$ is an eigenform for the Laplacian.

We recall the general formula ([AMP2])

$$\begin{aligned}\triangle(fF) = {} & [f(4\nu + \|\omega_2\|^2 + \|\omega_3\|^2) + \triangle f]F \\ & + [f(\delta\omega_3 - \langle\omega_1, \omega_2\rangle) - 2\langle df, \omega_3\rangle]F_2 \\ & + [-f(\delta\omega_2 + \langle\omega_1, \omega_3\rangle) + 2\langle df, \omega_2\rangle]F_3 \ ,\end{aligned}$$

where $H = (J_1 = J, J_2, J_3)$ is an admissible basis of Q. Since we have

$$df = \frac{1}{2} f\theta \ ,$$

$$\omega_2 = -\frac{1}{2}\theta \circ J_2 \quad , \quad \omega_3 = -\frac{1}{2}\theta \circ J_3 \ ,$$

and

$$\delta\omega_2 + \langle\omega_1, \omega_3\rangle = 0 \quad , \quad \delta\omega_3 - \langle\omega_1, \omega_2\rangle = 0 \ ,$$

(see [AMP2]), the formula reduces to

$$\triangle(fF) = [f(4\nu + \frac{1}{2}\|\theta\|^2) + \triangle f]F \ .$$

Moreover

$$\triangle f = \delta df = \frac{1}{2}\delta(f\theta) = \frac{1}{2}(-\langle df, \theta\rangle + f\delta\theta) = \frac{1}{4}f(2\delta\theta - \|\theta\|^2)$$

and by using the identity $\|\theta\|^2 + 2\delta\theta = 8n\nu$ proved in [AMP2]

$$\triangle f = f(2n\nu - \frac{1}{2}\|\theta\|^2) \ .$$

Hence by substitution in previous formula we get

$$\triangle(fF) = [f(4\nu + \frac{1}{2}\|\theta\|^2) + f(2n\nu - \frac{1}{2}\|\theta\|^2)]F \ ,$$

that is

$$\triangle\sigma = 2(n+2)\nu\sigma \qquad \text{or} \qquad \triangle\sigma = \frac{K}{2n}\sigma \ . \qquad \square$$

Remark. By refering to $\delta\sigma$ we recall that in [AM1] the following inequality for the first non-zero eigenvalue of Laplacian acting on 1-forms of a quaternion-Kähler manifold of positive scalar curvature was proved:

$$\lambda_1 \geq 2(n+1)\nu \qquad \text{that is} \qquad \lambda_1 \geq \frac{K}{2n} \cdot \frac{n+1}{n+2}$$

It is interesting to compare our last theorem with the following classical results of A. Lichnerowicz and Y. Matsushima, see [Be, pag. 330].

Theorem. *Let (M^{2m}, g, J) be a Kähler-Einstein manifold with scalar curvature $\overline{K} > 0$. Then the first non-zero eigenvalue λ_1 of Laplacian acting on functions satisfies the inequality*

$$\lambda_1 \geq \frac{\overline{K}}{m}$$

and the equality holds if and only if there exists an eigenfunction f such that the dual vector field $X = (df \circ J)^\sharp$ is Killing.

Let notice that the comparison between our theorem which we proved in [AMP3] for a quaternion-Kähler manifold and the above theorem valid for a Kähler-Einstein manifold is useful also by taking into account a classical result of S. Salamon and L. Bérard Bergery, see [Be pag. 416]: the twistor space Z of a quaternion-Kähler manifold (M^{4n}, g, Q) with positive scalar curvature K, which is a complex manifold of dimension $m = n + 1$, admits a Kähler-Einstein metric with positive scalar curvature $\overline{K}$ such that the projection $\pi : Z \longrightarrow M^{4n}$ is a Riemannian submersion with totally geodesic fibers.

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