Connections and Regularity on the Tangent and Cotangent Bundle

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Abstract

Considering some regular geometric objects on the tangent and cotangent bundles of a smooth manifold M, we define some diffeomorphisms between these bundles, which establish some relations between the mechanical structures on TM and T^*M. We construct a mechanical structure on T^*M and we also study the condition under which a regular vector field defining a mechanical structure on T^*M is a semi-Hamilton or Hamilton vector field. [2]

Introduction

There are some essential differences between the differential geometries of the tangent and cotangent bundles of a differentiable manifold M , although the bundles are isomorphic as vector bundles. For example, the tangent bundle has a naturally defined integrable almost tangent structure J and one can define the notion of a semispray (second order differential equation vector field) on it. From a semispray one can derive easily a (nonlinear) connection on the tangent bundle. The cotangent bundle T^*M has a naturally defined symplectic structure ω but one cannot define naturally something simmilar to the notion of almost tangent structure or semispray.

In [1] one introduces the notion of regularity and that of adapted almost tangent structure on the cotangent bundle T^*M, used for the study of some geometric properties of the tangent and cotangent bundles, considering some diffeomorphisms between them. A regular vector field K on T^*M defines an integrable adapted almost tangent structure and a connection Γ_K on T^*M. On the other hand, this regular vector field defines naturally an M-bundle diffeomorphism Φ from T^*M to TM which transfers some results from the differential geometry of the cotangent bundle to that of the tangent bundle, and conversely. Using a regular vector field (which always exists) one defines the notion of mechanical structure on T^*M , which generalizes the notion of Hamiltonian system on $T^\star M$ and is correlated with the notion of mechanical system on TM. A regular vector field K on $T^\star M$ defining a mechanical structure is used to define other two symmetric connections Γ'_K and Γ''_K on $T^\star M$. In this paper we study the conditions for the existence of regular vector fields K' and K'' on $T^\star M$, with the same associated M-vector field like K, such that the associated connections to K' and K'', $\Gamma_{K'}$ and $\Gamma_{K''}$ are just Γ'_K and Γ''_K.

In [2], based on a similar idea, considering a regular 1-form on TM we constructed a semispray which defines a connection on TM. The above regular 1-form defines

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naturally an M-bundle diffeomorphism Ψ from TM to T^*M, of the Legendre transformation type, which transfers some results from the differential geometry of the tangent bundle to that of the cotangent bundle, and conversely.

In [3] we established some relations between regular 1-forms and mechanical systems on TM.

Considering the diffeomorphism Ψ from TM to $T^\star M$, determined by a regular 1-form defining a mechanical systems on TM, we construct in this paper a mechanical structure on $T^\star M$, and also, we study the conditions under which the regular vector field defining a mechanical structure on $T^\star M$ is a semi-Hamilton or Hamilton vector field.

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1 Preliminaries

We begin by describing the geometric structures of the tangent and cotangent bundles TM and T^*M of a smooth $n-$dimensional manifold M. First, we shall present, in short, some result included in [1],[2]and [3]. Denote by $\tau : TM \longrightarrow M$, $\pi : T^*M \longrightarrow M$ the natural projections and let $(U, x^i), i = 1, ..., n$, be a local chart on M. Then the local chart $(\tau^{-1}(U); x^i, y^i)$ is induced on TM where $x^i = x^i \circ \tau$ (by abuse of notation) and y^i are the vector space coordinates with respect to the natural local frame $(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n})$ in TM defined by (U, x^i). Similarly the local chart $(\pi^{-1}(U); q^i, p_i)$ is induced on T^*M where $q^i = x^i \circ\pi$ and p_i are the vector space coordinates with respect to the natural local frame $(dx^1, ..., dx^n)$ in $T^\star M$ defined by (U, x^i).

Due to the special form of the change rules of the induced local charts on TM (on T^*M), it is possible to define the notion of an M-tensor field on these bundles. An M-tensor field of type (k, l) on TM (on $T^\star M$) is defined by sets of local coordinate components $T^{i_1...i_k}_{j_i...j_l}$, assigned to every induced local chart on TM (on $T^\star M$) such that the change rule, when a change of induced local charts on TM (on T^*M) is performed, is the same with the change rule of the local coordinate components of a tensor field of type (k, l) on M.

Consider the natural integrable almost tangent structure on TM, defined by the tensor field J of type $(1, 1)$ on TM having the local coordinate expression

$$J = \frac{\partial}{\partial y^i} \otimes dx^i.$$

In the case of the cotangent bundle the notion of adapted almost tangent structure is defined ([1]) by a tensor field J on $T^\star M$ of type (1,1) such that Ker J = Im $J = VT^*M$, where VT^*M = Ker π_* is the integrable vertical distribution. The local coordinate expression of J must be

$$J = g_{ij}\frac{\partial}{\partial p_i} \otimes dq^j \quad , \tag{1}$$

where the components $g_{ij}(q,p)$ define a nondegenerate M-tensor field of type (0,2) on $T^\star M$. The integrability condition for the adapted almost tangent structure consists in the existence of a regular M-vector field $\xi^i(q,p)$ (an M-tensor field of type (1,0)) on $T^\star M$ such that $g^{ij} = \frac{\partial \xi^j}{\partial p_i}$, where $g^{ij}(q,p)$ are the entries of the inverse of the matrix $(g_{ij}(q,p))$, i.e. $g_{ij}g^{ik} = g_{ji}g^{ki} = \delta^k_j$, and the matrix $(g_{ij}(q,p))$ is nondegenerate.

The components $\xi^i(q,p)$ of an M-vector field on $T^\star M$ may be thought of as the last n components of an M-bundle (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$. So, the local coordinate expression of Φ is

$$x^i = q^i \quad , \quad y^i = \xi^i(q,p). \tag{2}$$

The inverse Φ^{-1} of Φ has the local coordinate expression

$$q^i = x^i \quad , \quad p_i = \zeta_i(x,y). \tag{3}$$

The components $\zeta_i(x,y)$ define an M-1-form on TM.
Let K be a vector field on $T^\star M$, with the local coordinate expression

$$K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i} \ . \tag{4}$$

The components ξ^i define an M-vector field on $T^\star M$, called the associated M-vector field of K. The vector field K is called regular if its associated M-vector field is regular, i.e. the matrix $(\frac{\partial \xi^j}{\partial p_i})$ is nondegenerate. From a regular vector field K (from its associated M-vector field) one can derive an integrable adapted almost tangent structure J_ξ , with the local coordinate expression (1), and a nonlinear connection Γ_K, given by

$$\Gamma_K = -\mathcal{L}_K J_\xi \ . \tag{5}$$

Consider the canonical 1-form (the Liouville form) $\theta = p_i dq^i$ on $T^\star M$ and let $\omega = d\theta$ be the canonical symplectic structure on $T^\star M$. Recall that a vector field $H = \xi^i \frac{\partial}{\partial q^i} + \chi_i \frac{\partial}{\partial p_i}$ on $T^\star M$ is called a Hamilton vector field if it is regular and $\mathcal{L}_H \omega = 0$. This condition is expressed locally by the relations

$$(i)\frac{\partial \xi^i}{\partial p_j} = \frac{\partial \xi^j}{\partial p_i} \ , \quad (ii)\frac{\partial \chi_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial q^i} \ , \quad (iii)\frac{\partial \chi_i}{\partial q^j} = \frac{\partial \chi_j}{\partial q^i} \ . \tag{6}$$

By definition, a vector field K on $T^\star M$ defines a mechanical structure on $T^\star M$ if it is regular and the matrix $(g^{ij}) = (\frac{\partial \xi^j}{\partial p_i})$ defined by its associated M-tensor field is symmetric. In this case only the condition (6) (i) is fulfilled. There exists a smooth real function $h : T^*M \longrightarrow \mathbf{R}$ such that $\xi^i = \frac{\partial h}{\partial p_i}$ but, generally, $\chi_i \neq -\frac{\partial h}{\partial q^i}$. Consider the form $\eta = i_K\omega - (-dh)$. So $i_K\omega = -dh + \eta$. Therefore $\chi_i = -\frac{\partial h}{\partial q^i} + \eta_i$. Then $\eta = \eta_i dq^i$ is a semibasic 1-form on $T^\star M$.
The vector field K on T^*M is a semi-Hamilton vector field if it is regular and $\mathcal{L}_K\omega$ is a semibasic 2-form on $T^\star M$, i.e. $i_V(\mathcal{L}_K\omega) = 0$, whenever V is a vertical vector field on $T^\star M$. This condition may be expressed locally by (6) (i) and (6) (ii).
In the case of a vector field K defining a mechanical structure on $T^\star M$ one defines in

[1] another two connections on $T^\star M$. First, one defines the symmetric tensor field g of type (0,2) on $T^\star M$, by

$$g(Z_1, Z_2) = -\omega(Z_1, J_\xi Z_2) \qquad Z_1, Z_2 \in \chi(T^*M) \ .$$

Its local coordinate expression is $g = g_{ij} dq^i \otimes dq^j$. Then the nonlinear connection Γ'_K on $T^\star M$ is defined by the condition

$$\omega(Z_1, \Gamma'_K Z_2) = (\mathcal{L}_K g)(Z_1, Z_2) \qquad Z_1, Z_2 \in \chi(TM).$$

Another connection Γ''_K on $T^\star M$ defined by a vector field K defining a mechanical structure is obtained in a similar way from the tensor field $\tilde{g} = g_{ij} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}$. Both connections Γ'_K and Γ''_K are symmetric.
Recall that a semispray (a second order differential vector field) on the tangent bundle TM is a vector field S on TM such that $JS = C$, where C is the Liouville vector field on TM (the local coordinate expression is $C = y^i \frac{\partial}{\partial y^i}$). The local coordinate expression of a semispray is $S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x,y) \frac{\partial}{\partial y^i}$. There is a connection Γ (thought of as an almost product structure) on TM derived from the semispray S by

$$\Gamma = -\mathcal{L}_S J \ . \tag{7}$$

The connection Γ is called the canonical connection associated to the semispray S.

In [1] it is proved the following result: Let K be a regular vector field on $T^\star M$ and assume that its associated M-vector field defines a global M-bundle diffeomorphism $\Phi : T^*M \longrightarrow TM$.Then the vector field $\Phi_\star K$ is a semispray on TM whose derived connection Γ is just the image by Φ of the connection Γ_K defined by (5).

In [2] it is considered a regular 1-form on TM from which one derives, by using the integrable almost tangent structure and the Liouville vector field on TM, a semispray on TM. This semispray has its associated connection. Then, the above regular 1-form defines naturally an M-bundle (local) diffeomorphism Ψ from TM to $T^\star M$ by which we can establish some correspondences between the differential geometry of these bundles.

Let α be a 1-form on TM with local coordinate expression

$$\alpha = \alpha_j(x,y) dy^j + \beta_j(x,y) dx^j. \tag{8}$$

The components α_j define an M-1-form on TM, called the associated M-1-form of α. The form α is called regular if the matrix with the entries ($g_{ij} = \frac{\partial \alpha_i}{\partial y^j}$) is nondegenerate. Assuming that the 1-form α is regular, we consider the
1-form θ on TM defined by

$$\theta = \alpha \circ J = i_J \alpha \ \ , \tag{9}$$

where i_J is the vertical derivation on TM. The 1-form θ corresponds to the Cartan 1-form defined by a regular Lagrangian on TM and has the local coordinate expression

$$\theta = \alpha_i(x,y) dx^i \ .$$

The 2-form $d\theta = di_J\alpha$ is a symplectic form. Consider the 1-form β on TM defined by

$$\beta = di_C\alpha - \alpha \ , \tag{10}$$

and define the vector field S on TM by

$$i_S d\theta = -\beta \ . \tag{11}$$

We find that S is a semispray on TM . Its associated connection is given by (7). Just like in the case of a regular M-vector field, the components $\alpha_i(x, y)$ of α may be thought as the last n components of an M-bundle map $\Psi : TM \longrightarrow T^\star M$. This is the (local) diffeomorphism mentioned above. The local coordinate expression of Ψ is

$$q^i = x^i \quad , \quad p_i = \alpha_i(x, y) \ , \tag{12}$$

where $(\pi^{-1}(U); q^i, p_i)$ is the induced local chart on $T^\star M$. The inverse Ψ^{-1} of Ψ has the local coordinate expression

$$x^i = q^i \quad , \quad y^i = u^i(q, p) \ . \tag{13}$$

The central result of [2] is the following: If α is a regular 1-form on TM, there is a semispray S on TM derived from it by (9), (10), (11). Assuming that its associated M-1-form defines a global M-bundle diffeomorphism $\Psi : TM \longrightarrow T^\star M$, the vector field $\Psi_\star S$ is a regular vector field on $T^\star M$ whose derived connection Γ_K is the image by Ψ of the connection Γ associated to the semispray S.

2 Regular 1-forms and mechanical structures on TM and T^*M

In [3] there are studied some relations between regular 1-forms and regular mechanical systems on TM. In this section, considering the diffeomorphism $\Psi : TM \longrightarrow T^*M$ defined by a regular 1- form α, defining a mechanical system on TM, we shall construct a mechanical structure on $T^\star M$. Then, we shall study the condition under which the regular vector field defining a mechanical structure on $T^\star M$ is a semi-Hamilton vector field.

Recall that a mechanical system on TM is a triple $m = (M, F, \rho)$, where $F : TM \longrightarrow \mathbf{R}$ is a smooth real valued function and ρ is a semibasic 1-form on TM (i.e. $\rho(V) = 0$, for any vertical vector field V on TM). The vertical differentiation d_J on TM is a derivation of $\Lambda(M)$ of degree one, defined by

$$d_J = [i_j, d] = i_J d - di_J \ .$$

Consider the form $\omega_F = dd_J F$. If ω_F is a symplectic form (an equivalent condition is for the Hessian $(\frac{\partial^2 F}{\partial y^i \partial y^j})$ of F to be nondegenerate) the mechanical system m is said to be regular. In this case, the vector field S defined by

$$(14) \qquad i_S\omega_F = -dE_F + \rho \ ,$$

where $E_F = CF - F$ (the energy function associated to F) is a semispray.
m is said to be conservative if ρ is a closed form. In this case the mechanical system is Lagrangian. In [3] it is considered a regular 1-form $\alpha = \alpha_i(x,y)dy^j + \beta_i(x,y)dx^j$ such that its associated M-1-form defined by α_i satisfies the conditions

$$(15) \qquad \frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i} \ , \quad i,j = 1,...,n \quad (i.e.\ g_{ij} = g_{ij}) \ .$$

Then (15) may be expressed by

$$\alpha_i = \tfrac{\partial F}{\partial y^i} \ ,$$

where F is a smooth real valued function on TM.
Considering $\rho_i = \beta_i - \frac{\partial F}{\partial x^i}$, the form $\rho = \rho_i dx^i$ is a semibasic 1-form and $\alpha = dF + \rho$. The triple $m = (M, F, \rho)$ is proved to be a regular mechanical system and conversely, if $m = (M, F, \rho)$ is a regular mechanical system, then $\alpha = dF + \rho$ is a regular 1-form and its associated M-1-form satisfies (15).

Theorem 1 *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}$, $i = 1,...,n$, then the (local) diffeomorphism $\Psi : TM \longrightarrow T^*M$ given by* (12) *defines a mechanical structure $K = \Psi_* S$ on T^*M.*

Proof. By a straightforward computation we get

$$K = \Psi_* S = u^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i} \ .$$

From the condition for Ψ^{-1} to be the inverse of Ψ , we get $\frac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij}$. It follows

$$\tfrac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij} = g^{ji} = \tfrac{\partial u^i}{\partial p_j} \circ \Psi.$$

Therefore K defines a mechanical structure on $T^\star M$.

Proposition 1 *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}$, $i,j = 1,...,n$, then $\Psi_\star S$ is a semi-Hamilton vector field on $T^\star M$ if and only if the form ρ is a basic 1-form on TM.*
In other words, η is a basic 1-form on TM.

Proof. From the condition for Ψ^{-1} to be the inverse of Ψ , we also get

$$\tfrac{\partial u^j}{\partial q^k} \circ \Psi = -g^{ij}\tfrac{\partial \alpha_j}{\partial x^k} \ .$$

$\Psi_\star S$ is a semi-Hamiltonian vector field if and only if the condition (6) (ii) is fulfilled. In this case $\xi^i = u^i$ and $\chi_i = \beta_i \circ \Psi^{-1}$. Then the condition (6) (ii) becomes

$$\frac{\partial u^j}{\partial q^i} + \frac{\partial \beta_j}{\partial y^k}\frac{\partial u^k}{\partial p_j} = 0 \quad \Leftrightarrow \quad -g^{kj}\frac{\partial \alpha_k}{\partial x^i} + \frac{\partial \beta i}{\partial y^k}\, g^{jk} = 0 \quad \Leftrightarrow$$
$$g^{jk}(\frac{\partial \alpha_k}{\partial x^i} - \frac{\partial \beta_i}{\partial y^k}) = 0 \quad \Leftrightarrow \quad \frac{\partial \alpha_k}{\partial x^i} = \frac{\partial \beta_i}{\partial y^k} \quad \Leftrightarrow \quad \frac{\partial}{\partial y^k}(\frac{\partial F}{\partial x^i} - \beta_i) = 0 \quad \Leftrightarrow \quad \frac{\partial \rho_i}{\partial y^k} = 0.$$

The last conditions means that the form ρ is a basic 1-form on TM.

Proposition 2 *Under the hypothese of proposition 1 , the mechanical system defined by α on TM is conservative (Lagrangian) if and only if the mechanical system defined by $K = \Psi_\star S$ is conservative (Hamiltonian).*
In other words, ρ is a closed form on TM if and only if η is a closed form on $T^\star M$.

Proof. The mechanical system defined by $K = \Psi_\star S$ is conservative if and only if $d\eta = 0$, so $\mathcal{L}_K\omega = 0$. Taking into account the previous result, we have that (6) *(iii)* is equivalent with $\frac{\partial \beta_i}{\partial x^j} = \frac{\partial \beta j}{\partial x^i}$. But $\beta_i = \frac{\partial F}{\partial x^i} + \rho_i$. Therefore $\frac{\partial \beta_i}{\partial x^j} = \frac{\partial \beta j}{\partial x^i}$ *if and only if* $\frac{\partial \rho_i}{\partial x^j} = \frac{\partial \rho_j}{\partial x^i}$ and, since ρ is already a basic form this means that ρ is a closed form on TM.

3 Conditions for existence of regular geometric objects on T^*M

Consider now another problem related to the above correspondences between the tangent and the cotangent bundles. Assume that K is a regular vector field on $T^\star M$ and let $\beta_i(x,y); i = 1,...,n;$ be $n-$smooth real valued function on the domain $\tau^{-1}(U)$ of an induced local chart on TM. We ask for the local existence of a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (of a regular M-1-form α_i) such that $\Phi_* K$ is just the semispray derived from α.

Theorem 2 *Consider a regular vector field K on $T^\star M$ with the local coordinate expression (4) and let $\beta_i(x,y); i = 1,...,n$ be $n-$smooth real valued function on the domain $\tau^{-1}(U)$ of an induced local chart on TM. If the ordinary differential system on $\tau^{-1}(U)$*

$$\frac{dx^1}{y^1} = ... = \frac{dx^n}{y^n} = \frac{dy^1}{\frac{\partial \xi^1}{\partial q^k}(x,\zeta(x,y))y^k + \frac{\partial \xi^1}{\partial p_k}(x,\zeta(x,y))\chi_k(x,\zeta(x,y))} = ... \tag{16}$$
$$....... = \frac{dy^n}{\frac{\partial \xi^n}{\partial q^k}(x,\zeta(x,y))y^k + \frac{\partial \xi^n}{\partial p_k}(x,\zeta(x,y))\chi_k(x,\zeta(x,y))} = ...$$
$$...... = \frac{d\alpha_1}{\beta_1} = ... = \frac{d\alpha_n}{\beta n}$$

has n prime integrals $C^1, ..., C^n(x,y,\alpha)$ such that

$$\frac{D(C_1,....,C_n)}{D(y^1,....,y^n)} \neq 0; \ \frac{D(C_1,....,C_n)}{D(\alpha_1,....,\alpha_n)} \neq 0; \tag{17}$$

then there exists a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (so, with the last n components β_i) such that the semispray $\Phi_* K$ derives from α.

Proof. Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ and its inverse $\Phi^{-1} : TM \longrightarrow T^*M$ having the local coordinate expression (2) and (3), respectively. The semispray S derived from α has the local coordinate expression

$$S = y^i \frac{\partial}{\partial x^i} + g^{ki}(\beta_k - \frac{\partial \alpha_k}{\partial x^l} y^l) \frac{\partial}{\partial y^i} \ .$$

The relations $\Phi_* K = S$ are equivalent with

$$g^{ji}(\beta_j - \frac{\partial \alpha_j}{\partial x^k} y^k) = y^j (\frac{\partial \xi^i}{\partial q^j} \circ \Phi^{-1}) + (\chi_j g^{ji}) \circ \Phi^{-1} \ .$$

From these relations transvected by $g_{ki} = \frac{\partial \alpha_k}{\partial y^i}$ we get the quasilinear partial differential system

$$\frac{\partial \alpha_i}{\partial x^j} y^j + \frac{\partial \alpha_i}{\partial y^j} [y^k \frac{\partial \xi^j}{\partial q^k}(x, \zeta(x,y)) + \frac{\partial \xi^j}{\partial p_k}(x, \zeta(x,y)) \chi_k(x, \zeta(x,y))] = \beta_i \ , \tag{18}$$

whose characteristic system is just (16). The prime integrals $C_1, ..., C_n$ assure, using the implicite function theorem, the obtaining of the components $\alpha_i(x,y)$.

Remark. Under the hypothesis of the existence of this regular M-1-form α_i (of a regular form α) let us consider the (local) diffeomorphism $\Psi : TM \longrightarrow T^*M$, defined by the M-1-form α_i. Then $\Psi = \Phi^{-1}$ if and only if

$$\alpha_i(x,y) = \zeta_i(x,y), i = 1, ..., n.$$

Using the relations (18) we obtain the following result:

Proposition 3 *Let $K(q,p) = \xi^i(q,p) \frac{\partial}{\partial q^i} + \chi_i(q,p) \frac{\partial}{\partial p_i}$ be a regular vector field on T^*M and let $\beta_i(x,y); i = 1, ..., n;$ be n smooth real valued function on the domain $\tau^{-1}(U)$ of an induced local chart on TM. Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ defined by K. There exists a regular 1-form $\alpha = \alpha_j dy^j + +\beta_j dx^j$ (so, with the last n components β_i) such that the semispray $\Phi_* K$ is derived from α and Ψ is just Φ^{-1} (in this case $\Psi_* S = K$) if and only if the components $\zeta_i(x,y)$ satisfy the conditions :*

$$\frac{\partial \zeta_i}{\partial x^j} y^j + \frac{\partial \zeta_i}{\partial y^j} [y^k \frac{\partial \xi^j}{\partial q^k}(x, \zeta(x,y)) + \frac{\partial \xi^j}{\partial p_k}(x, \zeta(x,y)) \chi_k(x, \zeta(x,y))] = \beta_i \ . \tag{19}$$

Let us consider a regular vector field $K(q,p) = \xi^i(q,p) \frac{\partial}{\partial q^i} + \chi_i(q,p) \frac{\partial}{\partial p_i}$ defining a mechanical structure on $T^\star M$. From K one can derive the connection Γ_K (its associated connection) and, also, the connections Γ'_K and Γ''_K , as above. We ask for the existence of a regular vector field K' on $T^\star M$, with the same associated M-vector field (ξ^i) such that $\Gamma'_K = \Gamma_{K'}$, that is Γ'_K is just the connection associated to K' .Then, we shall consider the same problem for Γ''_K .For the study of these problems we shall use the following result of [1]:

If the components $\xi^i(q,p);\ i = 1, ..., n$ define a regular vector field on T^*M and Γ^* is a connection on T^*M then there exists a regular vector field K on $T^\star M$ with its associated M-vector field ξ^i and such that $\Gamma^* = \Gamma_K$ if and only if the following integrability conditions are fulfilled

$$\frac{\delta g_{kj}}{\delta q^i} - \frac{\delta g_{ki}}{\delta q^j} = g_{hi}\frac{\partial \Gamma_{kj}}{\partial p_h} - g_{hj}\frac{\partial \Gamma_{ki}}{\partial p_h} \ , \tag{20}$$

where $\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \Gamma_{ji}\frac{\partial}{\partial p_j}$, $i = 1, ..., n$, are local vector fields defining a local frame in the horizontal distribution $HT^\star M$, defined by Γ^*.

Theorem 3 *Let $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ be a regular vector field defining a mecanical structure on T^*M. There exists a regular vector field K' with the same associated M-vector field (ξ^i), such that $\Gamma'_K = \Gamma_{K'}$ if and only if the following integrability condition are fulfilled:*

$$\left(\frac{\partial g_{ki}}{\partial p_h} g_{lj} - \frac{\partial g_{kj}}{\partial p_h} g_{li}\right)\left(\frac{\partial \xi^l}{\partial q^h} + \frac{\partial \chi_h}{\partial p_l}\right) = 0 \ . \tag{21}$$

Proof. The coefficients Γ'_{ij} of the connection Γ'_K are given by

$$\Gamma'_{ij} = \tfrac{1}{2}\{K g_{ij} + g_{ik}\tfrac{\partial \xi^k}{\partial q^j} + g_{kj}\tfrac{\partial \xi^k}{\partial q^i}\}.$$

The result follows by a straightforward computation , using the relations (20).

Remark. If K is a semi-Hamilton vector field , the expression from the second parenthesis vanishes. Therefore, the relations (21) are fulfilled, in this case.

Theorem 4 *Let $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ be a regular vector field defining a mechanical structure on T^*M. There exists a regular vector field K'' with the same associated M-vector field (ξ^i), such that $\Gamma''_K = \Gamma_{K''}$ if and only if the following integrability conditions are fulfilled:*

$$\frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ki}}{\partial q^j} = g_{kl}\left(g_{hj}\frac{\partial^2 \chi_i}{\partial p_h \partial p_l} - g_{hi}\frac{\partial^2 \chi_j}{\partial p_h \partial p_l}\right) \ . \tag{22}$$

Proof. One uses the relations (20), where the coefficients Γ''_{ij} of the connection Γ''_K are given by

$$\Gamma''_{ij} = \tfrac{1}{2}\{K g_{ij} - g_{ik}\tfrac{\partial \chi_j}{\partial p_k} - g_{kj}\tfrac{\partial \chi_i}{\partial p_k}\} \ .$$

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