# Special Spinors and Contact Geometry

Andrei Moroianu[1]

**Abstract**

The aim of this note is to outline some new results obtained in contact geometry by means of spinorial methods and in particular to exhibit some interesting relations between (complex) contact structures and (Kählerian) Killing spinors.

## 1 Introduction

While the notion of a contact structure is well-known to most differential geometers, that of a Killing spinor (though intensively studied by physicists under the name of supersymmetry) remained, for a quite long time, neglected by mathematicians. Killing spinors came to be studied only after 1980, when Th. Friedrich [3] proved that they arise as the eigenspinors corresponding to the least possible eigenvalue of the Dirac operator on compact spin manifolds with positive scalar curvature. More precisely, we have the

**Theorem 1.1** (Friedrich, 1980) *Any eigenvalue $\lambda$ of the Dirac operator on a compact spin manifold $M^n$ with positive scalar curvature $S$ satisfies the inequality*

$$\lambda^2 \geq \frac{n}{4\,(n-1)} \inf_M S. \tag{1}$$

*Moreover, if the equality holds, then every eigenspinor $\psi$ corresponding to $\lambda$ is a real Killing spinor, i.e. satisfies the equation*

$$\nabla_X \psi = \alpha X \cdot \psi, \quad \forall X \in TM \quad (\alpha = -\frac{\lambda}{n}). \tag{2}$$

After several steps were made towards their classification by H. Baum, Th. Friedrich, R. Grunewald and I. Kath (these are presented in a unified manner in [2]), Killing spinors (or, properly speaking, manifolds carrying them) were finally classified by C. Bär [1], who made a very elegant use of the so-called cone construction. This is where contact structures come into the play, since Bär shows that, with some low-dimensional exceptions, all simply connected manifolds carrying Killing spinors are contact manifolds (or round spheres, in even dimensions). More precisely, if $M^{2m+1}$ ($m > 3$) carries Killing spinors, then $M$ is either Einstein-Sasakian or 3-Sasakian (for the definitions see [1] for example).

Using the explicit relations between Sasakian structures and Killing spinors given by Th. Friedrich and I. Kath [2], we gave a description in [11] of the splitting of the algebra of infinitesimal isometries of Einstein-Sasakian and 3-Sasakian manifolds, and furthermore proved the following rigidity result:

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**Theorem 1.2** *The only simply connected 3–Sasakian manifold $(M^7, g, \xi_i)$ possessing an infinitesimal isometry of unit length, other than the Sasakian vector fields, is the unit sphere $S^7$.*

Let us now turn our attention to the complex case, and recall the following

**Definition 1.1** (cf. [7]) *Let $M^{2m}$ be a complex manifold of complex dimension $m = 2k+1$. A complex contact structure is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:*

(i) *$\{U_i\}$ is an open covering of $M$.*

(ii) *$\omega_i$ is a holomorphic 1-form on $U_i$.*

(iii) *$\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0} M)$ is non vanishing at every point of $U_i$.*

(iv) *$\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$, where $f_{ij}$ is a holomorphic function on $U_i \cap U_j$.*

Our main result will be the classification of all Kähler-Einstein manifolds of positive scalar curvature admitting a complex contact structure. This goes roughly as follows: first of all, if $M^{4k+2}$ is a Kähler manifold admitting a complex contact structure, then we construct for $k$ odd a canonical spinor on $M$ and for $k$ even a canonical section of the spinor bundle associated to a suitable Spin$^c$ structure of $M$ (this idea - for $k$ odd - stems from K.-D. Kirchberg and U. Semmelmann, see [6]). We then prove that the constructed spinor is a *Kählerian Killing spinor* (we have to define this notion in the Spin$^c$ case) if the given Kähler metric on the manifold $M$ is also Einstein. The next step is to construct a canonical $S^1$ bundle $N$ over $M$, which is endowed with a Riemannian metric and a spin structure, such that the above constructed Kählerian Killing spinor on $M$ induces a Killing spinor on $N$. Finally, using Bär's classification of such manifolds and some further algebraic properties of the Killing spinor, we conclude that $N$ has to be 3-Sasakian and furthermore, by the naturality of the construction of $N$, we are also able to characterise $M$ geometrically.

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## 2 A short review on spin and Spin$^c$ geometry

We will first recall some basic facts about spin and Spin$^c$ structures.

Consider an oriented Riemannian manifold $(M^n, g)$. Let $P_{SO(n)}M$ denote the bundle of oriented orthonormal frames on $M$.

**Definition 2.1** *The manifold $M$ is called spin if the there exists a 2–fold covering $P_{Spin_n}M$ of $P_{SO(n)}M$ with projection $\theta : P_{Spin_n}M \to P_{SO(n)}M$ satisfying the following conditions :*

*i)* $P_{Spin_n}M$ *is a principal bundle over* $M$ *with structure group* $Spin_n$*;*

*ii) If we denote by* $\phi$ *the canonical projection of* $Spin_n$ *over* $SO(n)$*, then for every* $u \in P_{Spin_n}M$ *and* $a \in Spin_n$ *we have*

$$\theta(ua) = \theta(u)\phi(a).$$

The bundle $P_{Spin_n}M$ is called a spin structure. Representation theory shows that the complex Clifford algebra $Cl(n)$ has (up to equivalence) exactly one irreducible complex representation $\Sigma_n$ for $n$ even and two irreducible complex representations $\Sigma_n^{\pm}$ for $n$ odd. In the last case, these two representations are equivalent when restricted to $\mathrm{Spin}_n$, and this restriction is denoted by $\Sigma_n$. For $n$ even, there is a splitting of $\Sigma M$ with respect to the action of the volume element in $\Sigma_n := \Sigma_n^+ \oplus \Sigma_n^-$ and one usually calls elements of $\Sigma_n^+$ (resp. $\Sigma_n^-$) positive (resp. negative) half-spinors. For arbitrary $n$, $\Sigma_n$ is called the complex spin representation, and its associated vector bundle $\Sigma M$ is called the complex spinor bundle. Sections of $\Sigma M$ are called spinors. If $M$ is even-dimensional we denote by $\Sigma^{\pm}M$ the subbundles of $\Sigma M$ corresponding to $\Sigma_n^{\pm}$. If, with respect to the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, a spinor $\psi$ is written as $\psi = \ p_+ + \psi_-$, then its conjugate $\bar{\psi}$ is defined to be $\psi_+ - \psi_-$.

**Definition 2.2** *A* $Spin^c$ *structure on an oriented Riemannian manifold* $(M^n, g)$ *is given by a* $U(1)$ *principal bundle* $P_{U(1)}M$ *and a* $Spin_n^c$ *principal bundle* $P_{Spin_n^c}M$ *together with a projection* $\theta : P_{Spin_n^c}M \to P_{SO(n)}M \times P_{U(1)}M$ *satisfying*

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

*for every* $\tilde{u} \in P_{Spin_n^c}M$ *and* $\tilde{a} \in Spin_n^c$*, where* $\xi$ *is the canonical 2–fold covering of* $Spin_n^c$ *over* $SO(n) \times U(1)$*. The complex line bundle associated to* $P_{U(1)}M$ *is called the auxiliary bundle of the given* $Spin^c$ *structure.*

Recall that $\mathrm{Spin}_n^c = \mathrm{Spin}_n \times_{\mathbf{Z}_2} \mathrm{U}(1)$, and that $\xi$ is given by $\xi([u,a]) = (\phi(u), a^2)$, where $\phi : \mathrm{Spin}_n \to \mathrm{SO}(n)$ is the canonical 2–fold covering. The complex representations of $\mathrm{Spin}_n^c$ are obviously the same as those of $\mathrm{Spin}_n$; thus to every $\mathrm{Spin}^c$ manifold is associated a spinor bundle just as is the case for spin manifolds.

If $M$ is spin, the Levi–Civita connection on $P_{SO(n)}M$ induces a connection on the spin structure $P_{Spin_n}M$, and thus a covariant derivative on $\Sigma M$ denoted by $\nabla$. Similarly, if $M$ has a $\mathrm{Spin}^c$ structure, then every connection form $A$ on $P_{U(1)}M$ defines (together with the Levi-Civita connection of $M$) a covariant derivative on $\Sigma M$ denoted by $\nabla^A$.

Spin structures are special case of $\mathrm{Spin}^c$ structures, because of the following

**Lemma 2.1** *A* $Spin^c$ *structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection* $A$ *of the auxiliary bundle* $L$ *is flat, then under this identification* $\nabla^A$ *corresponds to* $\nabla$ *on the spinor bundles.*

**Proof.** Notice that the triviality of the auxiliary bundle implies that we can exhibit a global section of U(1) that we shall call $\sigma$. Denote by $P_{Spin_n}M$ the inverse image by $\theta$ of $P_{SO(n)}M \times \sigma$. It is straightforward to check that this defines a spin structure on $M$ and that the connection on $P_{Spin_n^c}M$ restricts to the Levi-Civita connection on $P_{Spin_n}M$ if $\sigma$ can be chosen to be parallel, i.e. if $A$ defines a flat connection.

Q.E.D.

Let $M$ be a Spin$^c$ manifold with auxiliary connection $A$. On $\Sigma M$ there is a canonical hermitian product $(.,.)$, with respect to which the Clifford multiplication by vectors (which arises via the Clifford representation) is skew–Hermitian:

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM,\ \psi, \varphi \in \Sigma M. \tag{3}$$

We now define the Dirac operator as the composition $\gamma \circ \nabla^A$, where $\gamma$ denotes the Clifford contraction. The Dirac operator can be expressed using a local orthonormal frame $\{e_1, \cdots, e_n\}$ as

$$D = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}.$$

Suppose now that $(M^{2m}, g, J)$ is a Kähler manifold. We define the twisted Dirac operator $\tilde{D}$ as

$$\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla^A_{e_i} = -\sum_{i=1}^{2m} e_i \cdot \nabla^A_{J(e_i)}.$$

It satisfies

$$\tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0. \tag{4}$$

We also define the complex Dirac operators $D_\pm := \frac{1}{2}(D \mp i\tilde{D})$, and (4) becomes

$$D_-^2 = D_+^2 = 0 \quad \text{and} \quad D^2 = D_+D_- + D_-D_+. \tag{5}$$

Consider a local orthonormal frame $\{X_\alpha, Y_\alpha\}$ such that $Y_\alpha = J(X_\alpha)$. Then $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar\alpha} = \frac{1}{2}(X_\alpha + iY_\alpha)$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$, and $D_\pm$ can be expressed as

$$D_+ = 2\sum_{\alpha=1}^{m} Z_\alpha \cdot \nabla^A_{Z_{\bar\alpha}} \quad , \quad D_- = 2\sum_{\alpha=1}^{m} Z_{\bar\alpha} \cdot \nabla^A_{Z_\alpha}. \tag{6}$$

A $k$-form $\omega$ acts on $\Sigma M$ by

$$\omega \cdot \Psi := \sum_{i_1 < \cdots < i_k} \omega(e_{i_1}, \cdots, e_{i_k})\, e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi\ .$$

With respect to this action, the Kähler form $\Omega$ (defined by $\Omega(X,Y) = g(X, JY)$) satisfies

$$\Omega = \frac{1}{2}\sum_{i=1}^{2m} J(e_i) \cdot e_i = -\frac{1}{2}\sum_{i=1}^{2m} e_i \cdot J(e_i). \tag{7}$$

For later use let us note that

$$\sum_{\alpha=1}^{m} Z_\alpha \cdot Z_{\bar\alpha} = -\frac{i}{2}\Omega - \frac{m}{2} \quad , \quad \sum_{\alpha=1}^{m} Z_{\bar\alpha} \cdot Z_\alpha = \frac{i}{2}\Omega - \frac{m}{2}, \tag{8}$$

where $Z_\alpha$ and $Z_{\bar\alpha}$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ as before.

The action of $\Omega$ on $\Sigma M$ yields an orthogonal decomposition

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle associated to the eigenvalue $i\,\mu_r = i\,(m - 2\,r)$ of $\Omega$. If we define $\Sigma_{-1}M = \Sigma_{m+1}M = \{0\}$, then

$$D_\pm \Gamma(\Sigma_r M) \subset \Gamma(\Sigma_{r\pm 1} M). \tag{9}$$

# 3 Relations between complex contact structures and spinors

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line subbundle $L_{\mathcal{C}} \subset \Lambda^{1,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$ and local sections $\omega_i$. It is easy to see that

$$\mathcal{D} := \{Z \in T^{1,0}M \mid \omega(Z) = 0, \ \forall \omega \in L_{\mathcal{C}}\}$$

is a codimension 1 maximally non–integrable holomorphic sub–bundle of $T^{1,0}M$, and conversely, every such bundle defines a complex contact structure. Condition (iii) in Definition 1.1 entails that $L_{\mathcal{C}}^{k+1}$ is isomorphic to $K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of $M$.

Suppose for a while that $k$ is even, say $k = 2l$. The collection $(U_i, \omega_i \wedge (\partial\omega_i)^l)$ defines a holomorphic line bundle $L_l \subset \Lambda^{2l+1,0}M$, and from the definition of $\mathcal{C}$ we easily obtain

$$L_l \cong L_{\mathcal{C}}^{l+1}. \tag{10}$$

We now fix some $(U, \omega) \in \mathcal{C}$ and define a local section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$ by

$$\psi_{\mathcal{C}}|_U := |\xi_\tau|^{-2} \bar\tau \otimes \xi_\tau, \tag{11}$$

where $\tau := \omega \wedge (\partial\omega)^l$ and $\xi_\tau$ is the element corresponding to $\tau$ through the isomorphism (10). The fact that $\psi_{\mathcal{C}}$ does not depend on the element $(U, \omega) \in \mathcal{C}$ shows that it actually defines a global section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$.

We now recall ([8], Appendix D) that $\Lambda^{0,*}M$ is just the spinor bundle associated to the canonical $\mathrm{Spin}^c$ structure on $M$, whose auxiliary line bundle is $K^{-1}$, so that $\Lambda^{0,*}M \otimes L_{\mathcal{C}}^{l+1}$ is actually the spinor bundle associated to the $\mathrm{Spin}^c$ structure on $M$ with auxiliary bundle $L = K^{-1} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}^{-(2l+1)} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}$. The section $\psi_{\mathcal{C}}$ is thus a spinor lying in $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1} \cong \Sigma_{2l+1}M$, which shows that

$$\Omega \cdot \psi_{\mathcal{C}} = -i\psi_{\mathcal{C}}. \tag{12}$$

The case $k = 2l+1$ is similar: the section $\psi_{\mathcal{C}}$ is defined by the same formulae as before, and it lies in $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1} \cong \Lambda^{0,2l+1}M \otimes K^{\frac{1}{2}}$. Thus in this case $\psi_{\mathcal{C}}$ is an usual spinor on $M$ (see[4]).

We suppose from now on that $M$ is Kähler-Einstein with positive scalar curvature. The manifold $M$ is compact, by Myers' Theorem. By rescaling the metric on $M$ if necessary, we can suppose that the scalar curvature of $M$ is equal to $2m(2m+2)$, and thus the Ricci form $\rho$ and the Kähler form $\Omega$ are related by the equality $\rho = (2m+2)\Omega$.

**Proposition 3.1** *For $k$ even the spinor field $\psi_{\mathcal{C}}$ satisfies*

$$\nabla_Z \psi_{\mathcal{C}} = 0, \ \forall Z \in T^{1,0}M \tag{13}$$

*and*

$$D^2\psi_{\mathcal{C}} = D_- D_+ \psi_{\mathcal{C}} = (\frac{1}{4} R\,\psi_{\mathcal{C}} - \frac{i}{2}\rho \cdot \psi_{\mathcal{C}}), \tag{14}$$

*and for $k$ odd, say $k = 2l+1$*

$$D^2\psi_{\mathcal{C}} = D_- D_+ \psi_{\mathcal{C}} = \frac{l+1}{2l+1}(\frac{1}{2} R\,\psi_{\mathcal{C}} - i\rho \cdot \psi_{\mathcal{C}}), \tag{15}$$

*where $R$ is the scalar curvature of $M$. In particular (13) shows that $D_-\psi_{\mathcal{C}} = 0$.*

The proof of the first two assertions can be found in [6]. The proof of (15) is analogous to that of (14) (one only has to replace some $\frac{1}{2}$ coefficients by $\frac{l+1}{2l+1}$ coefficients.

Using (12), (15) and the fact that $\rho = \frac{1}{8l+2} R\,\Omega = (8l+4)\Omega$ for $k = 2l$ and $\rho = \frac{1}{8l+6} R\,\Omega = (8l+8)\Omega$ for $k = 2l+1$, we obtain

**Corollary 3.1** *The spinor field $\psi_{\mathcal{C}}$ is an eigenspinor of $D^2$ with eigenvalue* $16l(l+1)$ *for $k = 2l$ and with eigenvalue* $16(l+1)^2$ *for $k = 2l+1$.*

It is now easy to see that for $k = 2l+1$ $\psi_{\mathcal{C}}$ is a Kählerian Killing spinor. Indeed, it is enough to use the above corollary and the fact that the scalar curvature of $M$ is (due to our normalisation) $S = 2m(2m+2) = (8l+6)(8l+8)$, together with the following result from [5]

**Theorem 3.1** (Kirchberg, 1986) *Any eigenvalue $\lambda$ of the Dirac operator on a compact Kähler spin manifold $(M^{2m}, g, J)$ ($m$ odd) with positive scalar curvature $S$ satisfies the inequality*

$$\lambda^2 \geq \frac{m+1}{4\,m} \inf_M S. \tag{16}$$

*Moreover, if the equality holds, then every eigenspinor $\psi$ corresponding to $\lambda$ is a Kählerian Killing spinor, i.e. satisfies the equation*

$$\nabla_X\psi = \alpha X\cdot\psi + \alpha J(X)\cdot\bar{\psi}, \quad \forall X\in TM \quad (\alpha = -\frac{\lambda}{2m+2}). \tag{17}$$

We thus have the

**Corollary 3.2** *For $k$ odd, the spinor $\psi_{\mathcal{C}}$ is a Kählerian Killing spinor.*

The case $k=2l$ is somewhat harder, since no analogue of Kirchberg's Theorem is known for $\mathrm{Spin}^c$ manifolds and we have to resort to an "ad-hoc" argument. Let us first introduce some notations:

$$\psi_- := \psi_{\mathcal{C}} \in \Gamma(\Sigma_{2l+1}M) \qquad , \qquad \psi_+ := \frac{1}{4l+4}D\psi_{\mathcal{C}} \in \Gamma(\Sigma_{2l+2}M) \qquad . \tag{18}$$

Integrating over $M$ we immediately obtain from Corollary 3.1

$$|\psi_-|^2_{L^2} = \frac{l+1}{l}|\psi_+|^2_{L^2}. \tag{19}$$

**Proposition 3.2** *The following relations hold*

$$\nabla_Z\psi_- = 0, \ \forall Z\in T^{1,0}M, \tag{20}$$

$$\nabla_{\bar{Z}}\psi_- + \bar{Z}\cdot\psi_+ = 0, \ \forall \bar{Z}\in T^{0,1}M, \tag{21}$$

$$\nabla_{\bar{Z}}\psi_+ = 0, \ \forall \bar{Z}\in T^{0,1}M, \tag{22}$$

$$\nabla_Z\psi_+ + Z\cdot\psi_- = 0, \ \forall Z\in T^{1,0}M. \tag{23}$$

**Proof.** The first relation is part of Proposition 3.1. In order to prove (21), let us consider the local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ introduced in Section 2: $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$, where $Y_\alpha = J(X_\alpha)$, and $\{X_\alpha, Y_\alpha\}$ is a local orthonormal frame of $TM$. From (20) we find $\nabla_{Z_{\bar{\alpha}}}\psi_- = \nabla_{X_\alpha}\psi_- = i\nabla_{Y_\alpha}\psi_-$, so using (8) and (18) gives

$$\begin{aligned}
0 &\leq \sum_{\alpha=1}^{m}|\nabla_{Z_{\bar{\alpha}}}\psi_- + Z_{\bar{\alpha}}\cdot\psi_+|^2 \\
&= \sum_{\alpha=1}^{m}|\nabla_{X_\alpha}\psi_-|^2 - 2\Re e\sum_{\alpha=1}^{m}(\psi_+, Z_\alpha\cdot\nabla_{Z_{\bar{\alpha}}}\psi_-) - \sum_{\alpha=1}^{m}(\psi_+, Z_\alpha\cdot Z_{\bar{\alpha}}\cdot\psi_+) \\
&= \frac{1}{2}|\nabla\psi_-|^2 - \Re e(\psi_+, D_+\psi_-) - \frac{1}{2}(\psi_+, (-i\Omega - m)\psi_+) \\
&= \frac{1}{2}|\nabla\psi_-|^2 - (4l+4)|\psi_+|^2 + \frac{1}{2}(4l+4)|\psi_+|^2.
\end{aligned}$$

The last expression is by construction a positive function on $M$, say $|F|^2$. Integrating over $M$ and using the generalised Lichnerowicz formula ([8], Appendix D), Corollary 3.1 and (19), we obtain

$$
\begin{aligned}
|F|^2_{L^2} &= \frac{1}{2}(\nabla^*\nabla\psi_-,\psi_-)_{L^2} - (4l+4)|\psi_+|^2_{L^2} + \frac{1}{2}(4l+4)|\psi_+|^2_{L^2} \\
&= \frac{1}{2}(D^2\psi_- - \frac{1}{4}R\psi_- + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_-,\psi_-)_{L^2} - (2l+2)|\psi_+|^2_{L^2} \\
&= |\psi_-|^2_{L^2}\Big(8l(l+1) - \frac{(8l+2)(8l+4)}{8} + \frac{i}{4}\frac{-i(8l+4)}{2l+1} - 2l\Big) = 0,
\end{aligned}
$$

thus proving that $F = 0$ and consequently (21). To check the last two equations one has to make use of the operator $\tilde{D}$. From $D_-\psi_- = 0$ we find

$$0 = \frac{1}{4l+4}D_+^2\psi_- = D_+\psi_+, \tag{24}$$

and so

$$\tilde{D}\psi_+ = -iD\psi_+. \tag{25}$$

Let us choose a local orthonormal frame $e_i$; using (3), (7), (18) and (25) we compute

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{n}|\nabla_{e_j}\psi_+ + \frac{1}{2}(e_j - iJ(e_j))\cdot\psi_-|^2 \\
&= |\nabla\psi_+|^2 - \Re e((D+i\tilde{D})\psi_+,\psi_-) \\
&\quad -\frac{1}{4}\sum_{j=1}^{n}((e_j+iJ(e_j))\cdot(e_j-iJ(e_j))\cdot\psi_-,\psi_-) \\
&= |\nabla\psi_+|^2 - 2\Re e(D\psi_+,\psi_-) + ((m-i\Omega)\cdot\psi_-,\psi_-) \\
&= |\nabla\psi_+|^2 - 8l|\psi_-|^2 + 4l|\psi_-|^2 := |G|^2
\end{aligned}
$$

Just as before, we compute the integral over $M$ of the positive function $|G|^2$, namely

$$
\begin{aligned}
|G|^2_{L^2} &= |\nabla\psi_+|^2_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= (\nabla^*\nabla\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= (D^2\psi_+ - \frac{1}{4}R\psi_+ + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= |\psi_+|^2_{L^2}\Big(16l(l+1) - \frac{(8l+2)(8l+4)}{4} + \frac{i}{2}\frac{-3i(8l+4)}{2l+1} - 4(l+1)\Big) = 0,
\end{aligned}
$$

thus proving $G = 0$. Consequently $\nabla_X\psi_+ + \frac{1}{2}(X - iJ(X))\cdot\psi_- = 0$ for all $X \in TM$, which is equivalent to (22) and (23).

Q.E.D.

The above proposition motivates the following

**Definition 3.1** *A section $\psi$ of the spinor bundle of a given Spin$^c$ structure on a Kähler manifold $(M^{8l+2}, g, J)$ satisfying*

$$\nabla^A_X \psi = \frac{1}{2} X \cdot \psi + \frac{i}{2} JX \cdot \bar{\psi}, \quad \forall X \in TM \tag{26}$$

*is called a Kählerian Killing spinor.*

Defining $\psi := \psi_+ - \psi_-$ we immediately obtain the

**Corollary 3.3** *Let $\mathcal{C}$ be a complex contact structure on a Kähler–Einstein manifold $(M^{8l+2}, g, J)$. Then the Spin$^c$ structure on $M$ with auxiliary bundle $L_{\mathcal{C}}$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$.*

# 4 The classification of positive Kähler-Einstein contact manifolds

Let us first recall the following results from the theory of projectable spinors:

**Theorem 4.1** ([10]) *Let $M$ be a compact Kähler manifold of positive scalar curvature and complex dimension $4l+3$. If $\Sigma M$ carries a Kählerian Killing spinor, then the principal $U(1)$ bundle $\bar{M}$ associated to any maximal root of the canonical bundle of $M$ admits a canonical spin structure carrying Killing spinors.*

**Theorem 4.2** ([12]) *Let $M$ be a compact Kähler manifold of positive scalar curvature and complex dimension $4l+1$ such that there exists a Spin$^c$ structure on $M$ with auxiliary bundle $L$ and spinor bundle $\Sigma M$ satisfying $L^{\otimes(2l+1)} \cong \Lambda^{4l+1,0}M$. If $\Sigma M$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$, then the principal $U(1)$ bundle $\bar{M}$ associated to any maximal root of the canonical bundle of $M$ admits a canonical spin structure carrying Killing spinors.*

We are now able to give the classification of positive Kähler-Einstein contact manifolds:

**Theorem 4.3** *The only Kähler-Einstein manifolds of positive scalar curvature admitting a complex contact structure are the twistor spaces of quaternionic Kähler manifolds of positive scalar curvature.*

The notion of the twistor space over a quaternionic Kähler manifold was introduced by S. Salamon in [13], where he proves that these twistor spaces all admit Kähler-Einstein metrics and complex contact structures. Our Theorem 4.3 is thus a converse of Salamon's result, and it should be noted that it was also recently proved by C. LeBrun [9] using rather different methods.

**Proof of Theorem 4.3.** Let $M^{4k+2}$ be a positive Kähler-Einstein contact manifold and let $\bar{M}$ be the principal U(1) bundle associated to any maximal root of the canonical bundle of $M$. From Corollaries 3.2 and 3.3 and Theorems 4.1 and 4.2 we deduce that $\bar{M}$ carries a *projectable* Killing spinor $\psi$. This spinor then induces a parallel spinor $\Psi$ on the cone $C\bar{M}$ over $\bar{M}$, which is a Kähler manifold (cf. [1], [10], [12]). Moreover, using the projectability of $\psi$ we can compute the action of the Kähler form of $C\bar{M}$ on $\Psi$ (see [10]) and obtain that $\Psi \in \Sigma_{k+1}C\bar{M}$. From C. Bär's classification [1] we know that the restricted holonomy group of $C\bar{M}$ is one of the following: $\mathrm{SU}(2k+2)$, $\mathrm{Sp}(k+1)$ or 0. The fixed points of the spin representation of $\mathrm{SU}(2k+2)$ lie in $\Sigma_0$ and $\Sigma_{2k+2}$, so since $\Psi$ is a parallel spinor in $\Sigma_{k+1}C\bar{M}$, the restricted holonomy group of $C\bar{M}$ cannot be equal to $\mathrm{SU}(2k+2)$. This implies that the universal covering of $C\bar{M}$ is hyperkähler, and thus that the universal covering of $\bar{M}$ is 3–Sasakian (see [1]). Actually, using the Gysin exact sequence we can easily deduce that $\bar{M}$ is simply connected (see [2], p.85). On the other hand, the unit vertical vector field $V$ on $\bar{M}$ defines a Sasakian structure (see[2]) and it is well known that any Sasakian structure on a 3–Sasakian manifold $P^{4k-1}$ of non-constant sectional curvature belongs to the 2–sphere of Sasakian structures. Indeed, the cone $CP$ over $P$ has restricted holonomy $\mathrm{Sp}(k)$, and since the centralizer of $\mathrm{Sp}(k)$ in $\mathrm{U}(2k)$ is just $\mathrm{Sp}(1)$, every Kähler structure on $CP$ must belong to the 2–sphere of Kähler structures of $CP$, which is equivalent to our statement.

Now, $\bar{M}$ is regular in the direction of $V$, so an old result of Tanno implies that it is actually a regular 3–Sasakian manifold (cf. [14]). It is then well known that the quotient of $\bar{M}$ by the corresponding SO(3) action is a quaternionic Kähler manifold of positive scalar curvature, say $N$, and that the twistor space over $N$ is biholomorphic to the quotient of $\bar{M}$ by each of the $S^1$ actions given by the Sasakian vector fields, so in particular to $M$, which is the quotient of $\bar{M}$ by the $S^1$ action generated by $V$.

Q.E.D.

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Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Ziegelstr. 13a
D–10099 Berlin