$CR-$structures on the Translated Sphere Bundle

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Abstract

In this paper there are found the condition under which we have induced $CR-$ structure on the translated sphere bundle thought of as hypersurface of TM (generalization of the results from [1], [5], [6], [9]).

(M,g) is a Riemannian manifold and $(TM,\bar{g})$ is its tangent bundle endowed with the Sasaki metric $\bar{g}$. The usual almost complex structure is determined by the endomorphism J; then $(TM,J,\bar{g})$ is almost Kählerian. [2]

1 The pseudoconvex CR-structure

Definition *A $CR-$structure on the differentiable manifold M, is defined by a real vector subbundle $R(M)$ in TM so that*

(i) $[X,Y]-[JX,JY]\in\Gamma(R(M))$; $\forall X,Y\in\Gamma(R(M))$

(ii) $N_J(X,Y)=[JX,JY]-[X,Y]-J\{[JX,Y]+[X,JY]\}=0$; $\forall X,Y\in\Gamma(R(M))$

where $\Gamma(R(M))$ denotes the set of cross-sections of $R(M)$.

Remark *In fact $R(M)$ is the decomplexification of a complex vector subbundle $H(M)$ in the complexification $T^cM=TM\otimes C$.*

The fundamental example of $CR-$structures is supplied by the real submanifolds of the complex manifolds.

The Levi form of a $CR-$manifold $(M,H(M))$ measures the deviation of the bracket $[X-iJX,Y+iJY]$, where $X,Y\in\Gamma(R(M))$, from having its values in $H(M)\oplus\overline{H(M)}$.

Definition *A $CR-$manifold of hypersurface type $(M,H(M))$ is pseudoconvex if its Levi form is nondegenerate.*

If we denote by η a local 1-form having the decomplexification $R(M)$ of $H(M)$ as its null bundle, i.e.

$$R(M)=\{X\in TM\mid \eta(X)=0\},$$

then the condition for $(M,H(M))$ to be pseudoconvex is expressed by the property of η to be a contact form, i.e. $\eta\wedge(d\eta)^n\neq 0$ on M.

If $(M,H(M))$ is a pseudoconvex $CR-$ structure with the contact form η, then we may obtain an associated almost contact structure (ϕ,ξ,η) where the vector field ξ and the tensor field of type $(1,1)$ ϕ are defined by the conditions:

$$\eta(\xi)=1\ ,\ i_\xi d\eta=0\ ,\phi X=J(X-\eta(X)\xi)\ ,\ \forall X\in\chi(M).$$

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Theorem ([8]): *The complex involutivity condition of $(M, H(M))$ is equivalent to:*

$$S = 0 \ ,$$

where S is the tensor field of type (1,2) on M, defined by:

$$S(X,Y) = N_\phi(X,Y) + d\eta(X,Y)\xi + \\ +\eta(X)\phi(\mathcal{L}_\xi\phi)Y - \eta(Y)\phi(\mathcal{L}_\xi\phi)X,$$

$\mathcal{L}$ is Lie derivative of ϕ with respect to ξ and N_ϕ is the Nijenhuis tensor field of ϕ, defined by:

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y].$$

2 The Translated Sphere Bundle.

Let (M, g) be a Riemannian manifold of dimension (n+1). Let TM be its tangent bundle and let $\bar{\pi} : TM \to M$ be the canonical projection. Let $(x^1, \ldots, x^{n+1})$ be the local coordinates on M and let $(y^1, \ldots, y^{n+1})$ be the fibre coordinates.

We consider on the manifold M a 1-form denoted by θ which is parallel with respect to the connection D (such a form exists in the case of cosymplectic manifolds). Then it follows that θ has a constant norm on M. If $\theta \neq 0$, we may take $\|\theta\| = 1$. Let $\alpha = \theta^\#$ the vector field on M, corresponding to θ by the "musical" isomorphism defined by g, i.e. $g(X, \alpha) = \theta(X)$, $\forall X \in \chi(M)$.

Consider the translated unit tangent bundle $T_1^\alpha M$ as the sphere tangent bundle with the center in the vertices of the vector field α, i.e. if $v \in TM$ then $v \in T_1^\alpha M \iff g(v - \alpha(\bar{\pi}(v)), v - \alpha(\bar{\pi}(v))) = 1$. If $v = y^i \frac{\partial}{\partial x^i}$ we conclude that the translated unit tangent bundle $\pi : T_1^\alpha M \to M$ is a hypersurface in TM given in local coordinates by:

$$g_{ij} y^i y^j - 2\theta_i y^i = 0.$$

Denote by $b^i = y^i - \alpha^i$ and by $l_i = g_{ij} b^j = g_{i0} - \theta_i$. Thus, we can write

$$N = b^i \frac{\partial}{\partial y^i} \ .$$

Making the computations we get

$$\xi = b^i \frac{\delta}{\delta x^i}; \quad \eta(\frac{\delta}{\delta x^i}) = l_i, \quad \eta(Y_i) = 0 \ , \quad \phi(\frac{\delta}{\delta x^i}) = Y_i, \quad \phi(Y_i) = -\frac{\delta}{\delta x^i} + l_i \xi \ ,$$

where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_{ij}^k y^j \frac{\partial}{\partial y^k}$ and $Y_i = \frac{\partial}{\partial y^i} - l_i N$ are the generators for $\chi(T_1^\alpha M)$.

It follows that $R(T_1^\alpha M)$ is defined by the equation $\eta = 0$.

We find the following expression for $d\eta$:

$$d\eta(Y_i, Y_j) = d\eta(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0 \ , d\eta(Y_i, \frac{\delta}{\delta x^j}) = g_{ij} - l_i l_j.$$

From these relations we conclude that we have a pseudoconvex hypersurface. We also notice that the considered almost contact structure is that associated.

Proposition *The Levi-Civita connection $\dot{\nabla}$ of G is given by the formulas:*

$$\begin{cases} \dot{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^k_{ij} \frac{\delta}{\delta x^k} - \frac{1}{2} R^k_{0ij} Y_k \ , \ \dot{\nabla}_{Y_i} Y_j = -l_j Y_i \ , \\ \dot{\nabla}_{\frac{\delta}{\delta x^i}} Y_j = \Gamma^k_{ij} Y_k - \frac{1}{2} R^k_{ij0} \frac{\delta}{\delta x^k} \ , \ \dot{\nabla}_{Y_i} \frac{\delta}{\delta x^j} = -\frac{1}{2} R^k_{ji0} \frac{\delta}{\delta x^k} \end{cases}$$

Proposition *The expression of the tensor field S is given by the formulas:*

$$\begin{cases} S(Y_i, Y_j) = (R^k_{0ji} + l_j R^k_{0i0} - l_i R^k_{0j0}) Y_k \ , \\ S(\frac{\delta}{\delta x^i}, Y_j) = (R^k_{0ij} + l_i R^k_{0j0} - l_j R^k_{0i0}) \frac{\delta}{\delta x^k} \ , \\ S(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = (R^k_{0ij} + l_i R^k_{0j0} - l_j R^k_{0i0}) Y_k \ . \end{cases} .$$

The complex involutivity condition can be written as

$$R^k_{0ji} + l_j R^k_{0i0} - l_i R^k_{0j0} = 0 \ ,$$

which implies, after straightforward computations, that the base manifold M has the constant sectional curvature c.

Having the 1-form θ parallel with respect to D on M, we conclude that M has $c = 0$. Thus we have the following

Theorem *The hypersurface $T^\alpha_1 M$ of TM is a $CR-$manifold if and only if the base manifold M is flat (locally Euclidean).*

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