The Nonlinear Connections and Harmonicity

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Abstract

The problems studied in this paper are concerned with the harmonicity of the canonical projection $\pi : TM \to M$, where (M, g) is a Riemannian space and TM is its tangent bundle, and conversely, the harmonicity of the vector fields $\xi \in \chi(M)$ thought of as maps from M to TM. We consider, on TM, a Riemannian metric of Sasaki type defined by means of an arbitrary nonlinear connection on TM. [2]

1 Introduction

The problems studied in this paper are concerned with the harmonicity of the canonical projection $\pi : TM \to M$, where (M, g) is a Riemannian space and TM is its tangent bundle, and conversely, the harmonicity of the vector fields $\xi \in \chi(M)$ thought of as maps from M to TM. Problems of this kind have been studied when TM is endowed with the Riemannian Sasaki metric (see [7] and [13]), with the Riemannian Cheeger-Gromoll metric (see [11]) or when TM is endowed with a pseudo-Riemannian metric of complete lift type g^c (see [10]). T. Ishihara, A. Sanini, V. Oproiu, T. Hangan, P.Piu had preoccupations with this field.

In this paper the main idea consists in the modification of the nonlinear connection on TM. We shall no longer use the nonlinear connection defined by the Levi-Civita connection ∇ of (M, g) but, more generally, we shall use an arbitrary nonlinear connection on TM and we shall consider a Riemannian metric of Sasaki type on TM defined by means of this nonlinear connection. This relaxation will lead us to interesting results, those obtained with the first nonlinear connection, defined by the Levi-Civita connection, being quite rigid. E.g., by using an appropriate nonlinear connection, we obtain that for any $\xi \in \chi(M)$ there is a Riemannian metric of Sasaki type on TM such that ξ is an harmonic map.

Acknowledgement

The author wishes to express his gratitude to professor V.Oproiu for many helpful talks and hints concerning the argument discussed in this paper.

2 The tangent bundle and the Sasaki metric

Let (M, g) be an n-dimensional, smooth, Riemannian manifold without boundary and let $\pi : TM \to M$ be its tangent bundle. We consider an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ which determines the differential structure of M and let $\overline{\mathcal{A}} = \{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}_{\alpha \in I}$ be

[1]Partially supported by the Grant 644/1997, Ministerul Educaţiei Naţionale.

[2]1991 AMS Mathematics Subject Classification 53C07, 53C20, 58E20.

Keywords and phrases: tangent bundle, harmonic maps, nonlinear connections, Sasaki metric.

the induced atlas which determines the differential structure of TM; $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n$, $\Phi_\alpha(v) = (\varphi_\alpha(p), y^i) = (x^i_\alpha(p), y^i)$, where $p = \pi(v) \in U_\alpha$ and $v = y^i \left(\frac{\partial}{\partial x^i_\alpha}\right)_p$. TM becomes an $2n$-dimensional, smooth manifold without boundary. From now on, we shall not use any longer the α index, so the local charts will be denoted simply by $(U, \varphi) = (U, x^i)$, $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), x^i, y^i)$. Denote by Γ^k_{ij} the Christoffel symbols of the metric g and by ∇ the Levi-Civita connection of g. On TM we have the vertical distribution VTM defined by $V_v(TM) = \ker \pi_{*,v}$, $v \in TM$. Remark that $V(TM)$ is an integrable distribution. We consider a nonlinear connection on TM defined by the distribution $H(TM)$ on TM, complementary to $V(TM)$, i.e. $H_v(TM) \oplus V_v(TM) = T_v(TM)$, $v \in TM$. For any induced local chart $(\pi^{-1}(U), x^i, y^i)$ we have a local adaptated frame in $H(TM)$ defined by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y)\frac{\partial}{\partial y^j}, \quad i = 1, \dots, n \ , \tag{1}$$

where the local functions $N^j_i(x,y)$ are the connection coefficients of the nonlinear connection defined by $H(TM)$. The vector fields $\left(\frac{\partial}{\partial y^i}\right)$, $i = 1, \dots, n$ define a local frame for the vertical distribution $V(TM)$. Let $\xi = \xi^i \frac{\partial}{\partial x^i}$ be a (local) vector field on M. We define the horizontal, respectively, the vertical lift of ξ by:

$$\xi^H = \xi^i \frac{\delta}{\delta x^i}, \quad \xi^V = \xi^i \frac{\partial}{\partial y^i}.$$

We have $\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i}$ and $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$.

The system of local 1-forms $(dx^i, \delta y^i)$ defines the dual frame of the frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, where

$$\delta y^i = dy^i + N^i_j dx^j \ . \tag{2}$$

Denote

$$R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k} = \frac{\delta}{\delta x^j}(N^i_k) - \frac{\delta}{\delta x^k}(N^i_j). \tag{3}$$

Remark that $R^i_{kj} = -R^i_{jk}$. We have the following commutation formulas:

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{kj}\frac{\partial}{\partial y^i}, \quad \left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right] = \frac{\partial N^i_j}{\partial y^k}\frac{\partial}{\partial y^i}, \quad \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0. \tag{4}$$

It follows that the horizontal distribution $H(TM)$ is integrable if and only if $R^i_{jk} = 0$. We define the Riemannian metric g^S of Sasaki type on TM by:

$$g^S(X^V, Y^V) = g(X,Y), \quad g^S(X^H, Y^V) = 0, \quad g^S(X^H, Y^H) = g(X,Y) \ . \tag{5}$$

If $g = g_{ij}dx^i dx^j$ is the expression in local coordinates of g, then g^S is locally given by

$$g^S = g_{ij}dx^i dx^j + g_{ij}\delta y^i \delta y^j \tag{6}$$

and the matrix of g^S is:

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix} . \tag{7}$$

It follows that g^S is a Riemannian metric on TM; the fibres of the horizontal and vertical distributions are orthogonal each other with respect to g^S, and $V_v(TM)$ and $H_v(TM)$ are isometric with T_pM, where $p = \pi(v)$. T_pM is a flat submanifold of (TM, g^S).

We have $\pi_*\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}$ and $\pi_*\left(\frac{\partial}{\partial y^i}\right) = 0$. Hence it follows:

Proposition 1 *The map $\pi : (TM, g^S) \to (M, g)$ is a Riemannian submersion.*

Proposition 2 *The Levi-Civita connection ${}^S\nabla$ of g^S is locally given by:*

$$\left\{\begin{array}{lcl} {}^S\nabla_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} & = & \dfrac{1}{2}\left(-\dfrac{\partial g_{ij}}{\partial x^k} + \dfrac{\partial N_k^l}{\partial y^j} g_{li} + \dfrac{\partial N_k^l}{\partial y^i} g_{lj}\right) g^{kh} \dfrac{\delta}{\delta x^h} , \\ {}^S\nabla_{\frac{\partial}{\partial y^i}} \dfrac{\delta}{\delta x^j} & = & -\dfrac{1}{2} R_{kj}^l g_{li} g^{kh} \dfrac{\delta}{\delta x^h} + \dfrac{1}{2}\left(\dfrac{\partial g_{ik}}{\partial x^j} - \dfrac{\partial N_j^l}{\partial y^k} g_{li} - \dfrac{\partial N_j^l}{\partial y^i} g_{lk}\right) g^{kh} \dfrac{\partial}{\partial y^h} , \\ {}^S\nabla_{\frac{\delta}{\delta x^i}} \dfrac{\partial}{\partial y^j} & = & \dfrac{1}{2}\left(\dfrac{\partial g_{jk}}{\partial x^i} - \dfrac{\partial N_i^l}{\partial y^k} g_{lj} + \dfrac{\partial N_i^l}{\partial y^j} g_{lk}\right) g^{kh} \dfrac{\partial}{\partial y^h} - \dfrac{1}{2} R_{ki}^l g_{lj} g^{kh} \dfrac{\delta}{\delta x^h} , \\ {}^S\nabla_{\frac{\delta}{\delta x^i}} \dfrac{\delta}{\delta x^j} & = & \Gamma_{ij}^h \dfrac{\delta}{\delta x^h} + \dfrac{1}{2} R_{ji}^h \dfrac{\partial}{\partial y^h} . \end{array}\right. \tag{8}$$

3 The harmonicity of the canonical projection $\pi : TM \to M$

Let (M, g) and (N, h) be two Riemannian manifolds and let $f : M \to N$ be a smooth map. Denote by ${}^N\Gamma_{\beta\gamma}^\alpha$ the Christoffel symbols of the metric h and by ${}^N\nabla$ the Levi-Civita connection of h. Let (U, x^i) be a local chart on M in the point $p \in M$ and let (V, u^α) be a local chart on N in $f(p)$; the indices $i, j, k, h, l, r, \dots$ run over the set $\{1, \dots, n\}$ and the indices $\alpha, \beta, \gamma, \dots$ run over the set $\{1, \dots, m\}$, where m is the dimension of the manifold N. The second quadratic form of f in p, denoted by $\beta(f)_p$, is given by:

$$\beta(f)_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\{f_{ij}^\alpha - \Gamma_{ij}^k f_k^\alpha +^N \Gamma_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma\right\} \frac{\partial}{\partial u^\alpha} , \tag{9}$$

where $f_k^\alpha = \frac{\partial f^\alpha}{\partial x^k}$, $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. The map f is totally geodesic if $\beta(f) = 0$. The form $\beta(f)$ is symmetric. The tension field $\tau(f)$ of f is given by:

$$\tau(f) = \text{trace } \beta(f) = g^{ij} \beta(f)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) . \tag{10}$$

The map f is harmonic map if $\tau(f) = 0$ (see [2]). If $X \in \chi(M)$ is f-related with $\overline{X} \in \chi(N)$ i.e. $f_{*,p}X = \overline{X}(f(p)), \forall p \in M$, and $Y \in \chi(M)$ is f-related with $\overline{Y} \in \chi(N)$, then:

$$\beta(f)_p(X, Y) = \left({}^N\nabla_{\overline{X}} \overline{Y}\right)_{f(p)} - f_{*,p}(\nabla_X Y) \tag{9'}$$

Let us study the above notions in the case of the map $\pi : (TM, g^S) \to (M, g)$. From the relations (8) and (9') we obtain:

$$(11)\qquad \begin{cases} \beta(\pi)\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) & = \dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial N^l_k}{\partial y^j} g_{li} - \dfrac{\partial N^l_k}{\partial y^i} g_{lj}\right) g^{kh} \dfrac{\partial}{\partial x^h} \ , \\ \beta(\pi)\left(\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta x^j}\right) & = \dfrac{1}{2} R^l_{kj} g_{li} g^{kh} \dfrac{\partial}{\partial x^h}, \qquad \beta(\pi)\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) = 0 \ , \\ \tau(\pi) & = \dfrac{1}{2} g^{ij}\left(\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial N^l_k}{\partial y^j} g_{li} - \dfrac{\partial N^l_k}{\partial y^i} g_{lj}\right) g^{kh} \dfrac{\partial}{\partial x^h} \ . \end{cases}$$

Then, from the theory of Riemannian submersions or from the relations (11) and using the local coordinates we obtain:

Theorem 1 *For $\pi : (TM, g^S) \to (M, g)$ we have:*

- **a)** *The projection π has all its fibres totally geodesic if and only if $\beta(\pi)(X^V, Y^V) = 0$.*
- **b)** *The projection π is harmonic map if and only if it has all its fibres minimal submanifolds.*

Corollary. *$\pi : (TM, g^S) \to (M, g)$ is totally geodesic if and only if $H(TM)$ is integrable and π has all its fibres totally geodesic.*

Remark.

- **1)** If π has its fibres totally geodesic then π is harmonic.
- **2)** The conditions: $H(TM)$ is integrable and π has its fibres totally geodesic are independent:
  - if $N^i_j(x, y) = \left(\Gamma^i_{jl} - \frac{\partial f}{\partial x^j}\delta^i_l\right) y^l$ with $f \in \mathcal{F}(M)$ and (M, g) is flat, then π does not have its fibres totally geodesic but $H(TM)$ is integrable;
  - if $N^i_j(x, y) = \Gamma^i_{jl} y^l$ and (M, g) is not flat then π has its fibres totally geodesic but $H(TM)$ is not integrable.

From now on we shall analyse some examples of nonlinear connections whose expressions in local charts are polynomials in y^i. Consider

$$A)\qquad N^i_j(x, y) = R^i_{ljh} y^l y^h + \Gamma^i_{jl} y^l + T^i_j(x) \ ,$$

where R^i_{ljh} are the components of the curvature tensor R of ∇ and T^i_j are the components of a tensor field of type $(1, 1)$ on M.

Proposition 3

- **a)** *π has its fibres totally geodesic if and only if (M, g) is flat, i.e. $R = 0$.*
- **b)** *π is harmonic map if and only if (M, g) is Ricci-flat, i.e. $\rho(R) = 0$.*

Remark. If (M,g) is flat, then π has its fibres totally geodesic. If (M,g) has the sectional curvature constant $a \neq 0$, then π cannot be harmonic map.

$B)$
$$N^i_j(x,y) = \left(\Gamma^i_{jl} + B^i_{jl}(x)\right) y^l + T^i_j(x) \ ,$$

where $B^i_{jk}(x)$ are the components of a tensor field of type $(1,2)$ on M.

Proposition 4

a) *π has its fibres totally geodesic if and only if $B_{ikj}+B_{jki}=0$, where $B_{ikj} = g_{hi}B^h_{kj}$;*

b) *π is harmonic map if and only if $g^{ij}B_{ikj}=0$.*

Remark.

1) If $B_{ikj} = \nabla_i P_{kj} - \nabla_j P_{ki}$, or if $B_{ikj} = A_k P_{ij}$ and $P_{ij} = -P_{ji}$, then π has its fibres totally geodesic.

2) If $B_{ikj} = R^t_{kij}\xi_t$, then π has its fibres totally geodesic; if $B_{ikj} = \nabla_k\nabla_i\xi_j$ and ξ is a Killing vector field, then π has its fibres totally geodesic.

3) If $B_{ikj} = \nabla_i\nabla_j\xi_k$, M is flat, ξ – Killing or $\Delta\xi = 0$, then π is harmonic map; $\Delta\xi = 0$ i.e. $g^{ij}\nabla_i\nabla_j\xi^k = R^k_t\xi^t$, where R_{jk} is the local coordinate expression of the Ricci tensor field $\rho(R)$ and $R^i_k = R_{jk}g^{ji}$.

$B')$
$$N^i_j(x,y) = (\Gamma^i_{jl} - A_j\delta^i_l)y^l,$$

where A_j are the components of a 1-form A on M. By a straightforward computation we obtain:

$$R^i_{kj} = R^i_{lkj}y^l - F_{kj}y^i \ ,$$

where $F_{kj} = \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j}$.

Proposition 5 *$H(TM)$ is involutive if and only if $R^i_{lkj}=0$ and $F_{kj}=0$.*

Proof. We have $R^i_{lkj}y^l - F_{kj}y^i = 0 \iff R^i_{lkj} - F_{kj}\delta^i_l = 0 \iff R_{hlkj} = F_{kj}g_{lh}$. From $R_{hlkj} = F_{kj}g_{lh}$ we obtain $0 = nF_{kj} \Rightarrow F_{kj} = 0$. So $R^i_{kj} = 0$ if and only if $R^i_{lkj} = 0$ and $F_{kj} = 0$.

Remark. If $A_j = \frac{\partial f}{\partial x^j}$ where $f \in \mathcal{F}(M)$, then $F_{kj} = 0$.

Proposition 6 *We have $\beta(\pi)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}A^h\frac{\partial}{\partial x^h}$, where $A^h = g^{hi}A_i$. So π has its fibres totally geodesic if and only if $A = 0$.*

Theorem 2 *For any $\xi \in \chi(M)$ there is a Riemannian metric on TM, of type (5), such that $\pi : (TM, g^S) \to (M,g)$ has its tension field: $\tau(\pi) = \xi \circ \pi$.*

Proof. This results from the Proposition 6 considering $A_h = \frac{1}{n}\xi_h$.

$B'')$.
$$N^i_j(x,y) = \Gamma^i_{jl}y^l - T^i_j$$

By a straightforward computation we obtain:

$$R^i_{kj} = R^i_{lkj}y^l + (\nabla_j T^i_k - \nabla_k T^i_j) \ .$$

Proposition 7 *$H(TM)$ is integrable if and only if (M,g) is flat and T is Codazzi tensor field, i.e. $\nabla_j T^i_k = \nabla_k T^i_j$.*

Remark.

1) If $\nabla_k T_{ij} = 0$, then T is Codazzi tensor field. If $T_{ij} = R_{ij}$ and $\nabla R = 0$ i.e. (M,g) is locally symmetric, then $\nabla_k T_{ij} = 0$. If $T_{ij} = R_{ij}, (M,g)$ is Einstein manifold and $n > 2$, then $\nabla_k T_{ij} = 0$.

2) If $T_{ij} = \nabla_i A_j$, A-Killing, then $H(TM)$ is integrable if and only if (M,g) is flat.

3) If $T^i_j = \nabla_j \xi^i$, then $H(TM)$ is integrable if and only if (M,g) is flat.

Theorem 3 *We have:*

a) *π has its fibres totally geodesic;*

b) *π is harmonic map;*

c) *π is totally geodesic map if and only if $H(TM)$ is integrable.*

$C)$
$$N^i_j(x,y) =^c \Gamma^i_{jl}y^l \ ,$$

where ${}^c\Gamma^i_{jl}$ are the Christoffel symbols of a Riemannian metric c. We obtain $R^i_{kj} = {}^cR^i_{lkj}y^l$, where ${}^cR^i_{lkj}$ are the components of the curvature tensor ${}^cR$ of c.

Proposition 8 *$H(TM)$ is integrable if and only if (M,c) is flat, i.e. ${}^cR = 0$.*

Theorem 4

a) *$\pi : (TM, g^S) \to (M,g)$ has its fibres totally geodesic if and only if ${}^c\nabla g = 0$, where ${}^c\nabla$ is the Levi-Civita connection of c;*

b) *$\pi : (TM, g^S) \to (M,g)$ is harmonic map if and only if $g^{ij}({}^c\nabla_k g_{ij}) = 0$.*

We remark that: ${}^c\nabla g = 0 \iff {}^c\Gamma^i_{jk} = \Gamma^i_{jk} \iff \nabla c = 0$.

Remark. In the cases $B)$, $B')$, $B'')$ and $C)$ from above π has its tension field basic i.e. if $\pi(v_1) = \pi(v_2)$ then $\tau(\pi)(v_1) = \tau(\pi)(v_2)$. In the case $A)$ we obtain: $\tau(\pi) = R^t_l y^l \frac{\partial}{\partial x^t}$; if (M,g) is Einstein manifold, then $\tau(\pi) = \lambda 1$, where $\lambda \in \mathcal{F}(M)$, $1 : TM \to TM$ is the identity map. Hence it results that π has not its tension field basic.

4 The harmonicity of the vector fields $\xi \in \chi(M)$

In what folows, we shall study the harmonicity of the vector fields $\xi \in \chi(M)$ regarded as maps from M to TM.

Theorem 5 *Let (M, g) be a Riemannian manifold and G a Riemannian metric on TM. Assume that $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion and isometric immersion. Then ξ is a totally geodesic map.*

Proof. We have $V(TM) = \ker \pi_*$. Let $H(TM)$ be the orthogonal complement of $V(TM)$ with respect to G. Denote by $N^i_j(x, y)$ the connection coefficients of the non-linear connection defined by G. The projection $\pi : (TM, G) \to (M, g)$ is Riemannian submersion, so $G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g_{ij}$. We have

$$\xi_{*,p}\left(\frac{\partial}{\partial x^i}\right) = \frac{\delta}{\delta x^i} + \left(N^j_i \circ \xi + \frac{\partial \xi^j}{\partial x^i}\right)\frac{\partial}{\partial y^j}.$$

Because ξ is isometric immersion we have $N^j_i \circ \xi + \frac{\partial \xi^j}{\partial x^i} = 0$. So $\xi_{*,p}(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}(\xi(p))$. Let ${}^G\nabla$ be the Levi-Civita connection of G. We have:

$${}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \Gamma^k_{ij}\frac{\delta}{\delta x^k} + \frac{1}{2}R^k_{ji}\frac{\partial}{\partial y^k}\ ,$$

$$\begin{aligned}\beta(\xi)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= \left({}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}\right) \circ \xi - \xi_*\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right) = \Gamma^k_{ij}\frac{\delta}{\delta x^k} + \frac{1}{2}(R^k_{ji} \circ \xi)\frac{\partial}{\partial y^k} - \\ &\quad -\Gamma^k_{ij}\frac{\delta}{\delta x^k} = \frac{1}{2}(R^k_{ji} \circ \xi)\frac{\partial}{\partial y^k}\ .\end{aligned}$$

Using that $R^k_{ji}(\xi(p))\frac{\partial}{\partial y^k} = \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right](\xi(p)) = \xi_{*,p}\left(\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\right) = 0$ it results $\beta(\xi) = 0$, i.e. ξ is totally geodesic map.

The map $\xi : M \to TM$ is an one-one immersion. If we suppose that M is compact, then ξ is an embedding, so that $\xi(M)$ is a compact submanifold of TM.

Theorem 6 *If $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion, $\xi : (M, g) \to (TM, G)$ is an isometric immersion and M is compact, then $\xi(M)$ is a totally geodesic submanifold of (TM, G).*

Proof. We have

$$\begin{aligned}T_{\xi(p)}(\xi(M)) &= \text{spann}\left\{\xi_{*,p}\left(\frac{\partial}{\partial x^i}\right)\right\} = \text{spann}\left\{\frac{\delta}{\delta x^i}\xi(p)\right\} = H_{\xi(p)}(TM)\ , \\ T^{\perp}_{\xi(p)}(\xi(M)) &= \text{spann}\left\{\frac{\partial}{\partial y^i}\xi(p)\right\} = V_{\xi(p)}(TM)\ .\end{aligned}$$

Denote by B the second fundamental form of the submanifold $\xi(M)$. We have:

$$B_{\xi(p)}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \left({}^G\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}\right)^{\perp}_{\xi(p)} = \frac{1}{2}R^k_{ji}(\xi(p)) = 0\ .$$

So $\xi(M)$ is totally geodesic submanifold of (TM, G).

From now, we shall consider $N^i_j(x,y) = \Gamma^i_{j0} - T^i_j(x)$ and the Riemannian metric g^S of type (5).

We have

$$\xi_{*,p}(X) = \left\{X^H + (\nabla_X \xi - TX)^V\right\}(\xi(p)) \ . \tag{12}$$

Proposition 9 *The map $\xi : (M,g) \to (TM, g^S)$ is an isometric immersion if and only if*

$$\nabla_X \xi = TX, \quad \forall X \in \chi(M). \tag{13}$$

Proof. $g^S(\xi_* X, \xi_* Y) = g(X,Y) + g(\nabla_X \xi - TX, \nabla_Y \xi - TY)$. So ξ is an isometric immersion, i.e. $g(X,Y) = g^S(\xi_* X, \xi_* Y)$ if and only if $\nabla_X \xi = TX, \quad \forall X \in \chi(M)$.

Under the condition of the complete integrability for the partial differential system defined by (13), (M,g) must be flat and the tensor field T must be the Codazzi tensor field.

Remark.

1) For any $\xi \in \chi(M)$ if we choose $T^i_j = \nabla_j \xi^i$, we obtain ξ-isometric immersion.

2) If $\nabla \xi = 0$ and $T \neq 0$, then ξ is not an isometric immersion.

Proposition 10 *Let ${}^1G$ and ${}^2G$ be Riemannian metrics on TM. Suppose that $\pi : (TM, {}^1G) \to (M,g)$ and $\pi : (TM, {}^2G) \to (M,g)$ are Riemannian submersions and they determine the same nonlinear connection on TM.*

$\xi : (M,g) \to (TM, {}^1G)$ is an isometric immersion if and only if $\xi : (M,g) \to (TM, {}^2G)$ is an isometric immersion.

From the Theorems 5 and 6 we obtain:

Theorem 7

a) *For any $\xi \in \chi(M)$, there is a Riemannian metric g^S on TM of type (5), such that $\xi : (M,g) \to (TM, g^S)$ is a totally geodesic map.*

b) *Suppose that M is compact manifold. For any $\xi \in \chi(M)$ there is a Riemannian metric g^S on TM of type (5), such that $\xi(M)$ is a totally geodesic submanifold of (TM, g^S).*

Proposition 11 *The Levi-Civita connection ${}^S\nabla$ of g^S, for $N^i_j(x,y) = \Gamma^i_{j0} - T^i_j(x)$, is locally given by:*

$$\left\{\begin{array}{lcl} {}^S\nabla_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} & = & 0, \quad {}^S\nabla_{\frac{\partial}{\partial y^i}} \dfrac{\delta}{\delta x^j} = \dfrac{1}{2} R^l_{jk} g_{li} g^{kh} \dfrac{\delta}{\delta x^h} \ , \\ {}^S\nabla_{\frac{\delta}{\delta x^i}} \dfrac{\delta}{\delta x^j} & = & \Gamma^h_{ij} \dfrac{\delta}{\delta x^h} - \dfrac{1}{2} R^h_{ij} \dfrac{\partial}{\partial y^h}, \quad {}^S\nabla_{\frac{\delta}{\delta x^i}} \dfrac{\partial}{\partial y^j} = \Gamma^h_{ij} \dfrac{\partial}{\partial y^h} + \dfrac{1}{2} R^l_{ik} g_{lj} g^{kh} \dfrac{\delta}{\delta x^h} \ , \end{array}\right. \tag{14}$$

where $R^i_{kj} = R^i_{lkj} y^l + (\nabla_j T^i_k - \nabla_k T^i_j)$.

From the relations (12) and (14) we obtain:

$$\tau(\xi) = \left\{g^{ij} A_j^r (R_{ik}^l \circ \xi) g_{lr} g^{kh}\right\} \frac{\delta}{\delta x^h} + \left\{g^{ij} (\nabla_i A_j^h)\right\} \frac{\partial}{\partial y^h} , \tag{15}$$

where $A_i^k = \nabla_i \xi^k - T_i^k$.

From (15) we have:

Theorem 8 *$\xi : (M, g) \to (TM, g^S)$ is a harmonic map if and only if*

$$\begin{cases} g^{ij} A_j^r (R_{ik}^l \circ \xi) g_{lr} g^{kh} &= 0 , \\ g^{ij} (\nabla_i A_j^h) &= 0 . \end{cases}$$

Since any totally geodesic map is an harmonic map, from Theorem 7, we obtain:

Proposition 12

a) *For any $\xi \in \chi(M)$ there is a Riemannian metric g^S on TM, such that $\xi : (M, g) \to (TM, g^S)$ is an harmonic map.*

b) *If M is compact, then for any $\xi \in \chi(M)$ there is a Riemannian metric g^S on TM, such that $\xi(M)$ is a minimal submanifold of (TM, g^S).*

Proof. We choose $T_j^i = \nabla_j \xi^i$. From the relation (15) we obtain:

Proposition 13

a) *If $\nabla_i T_k^j = 0$, (M, g) is flat, ξ-Killing or $\Delta\xi = 0$, then ξ is an harmonic map.*

b) *If $\nabla_i T_k^j = 0, \nabla_i \xi^j = 0$, then ξ is an harmonic.*

c) *If M is compact and orientable, $\nabla_i T_k^j = 0$, then ξ is an harmonic map if and only if $\nabla\xi = 0$.*

Proof Indeed, if M is compact and orientable we have

$$\int_M \left\{g^{ij} (\nabla_i \nabla_j \xi^k) g_{kh} \xi^h + (\nabla_j \xi^i) g^{jr} (\nabla_r \xi^l) g_{il}\right\} \mathrm{d}v_g = 0 \quad \text{(see[14])} .$$

So, if $g^{ij}(\nabla_i \nabla_j \xi^k) = 0$ it follows $\nabla_i \xi^j = 0$.

Remark. Suppose that M is compact and orientable. If $T_j^k = 0$, then $\xi : (M, g) \to (TM, g^S)$ is an isometric immersion if and only if $\xi : (M, g) \to (TM, g^S)$ is an harmonic map. If $\nabla_i T_j^k = 0$ and $T \neq 0$, then ξ harmonic map not implies ξ isometric immersion.

Proposition 14 *We consider $T_i^j = \nabla_i \eta^j, \eta \in \chi(M)$. We have:*

a) *If $\nabla\eta = \nabla\xi$ i.e. $\xi = \eta + \theta$ with $\nabla\theta = 0$, then $\xi : (M, g) \to (TM, g^S)$ is an isometric immersion.*

b) *If M is compact and orientable, then $\xi : (M, g) \to (TM, g^S)$ is harmonic if and only if $\nabla\eta = \nabla\xi$.*

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