Some Kähler Structures on the Tangent Bundle of a Space Form

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Abstract

Some Kähler structures are obtained on the tangent bundle of a Riemannian manifold of constant sectional curvature. If the constant sectional curvature is negative then the found Kähler structure is defined on the whole tangent bundle and is Kähler Einstein. If the constant sectional curvature is positive then a Kähler Einstein structure is defined in a tube around zero section in the tangent bundle. Moreover, the holomorphic sectional curvature of the obtained Kähler Einstein structure is constant. Two special cases are also studied. [2]

Introduction

The tangent bundle TM of an $n-$dimensional smooth manifold M can be organized as a $2n-$dimensional manifold by using induced local charts from local charts on the base manifold. There exists a naturally defined integrable almost tangent structure on TM and it can be defined the notion of semispray (a vector field defining a second order differential system on M). Since TM is a natural vector bundle over M, it is possible to define several lifts of functions, vector fields, $1-$forms and, more generally, of tensor fields and linear connections defined on the base manifold, to similar objects defined on the total space of the tangent bundle TM. We have the vertical and complete lifts and, if the base manifold is endowed with a linear connection, then we can define also the horizontal lift with respect to this connection.

In this paper we shall present some interesting Kähler structures defined on the tangent bundle of a Riemannian manifold of constant sectional curvature obtained by adapting an idea of Calabi ([2], [1]) to define a hyperkaehler structure on the cotangent bundle of a Kähler manifold having constant positive holomorphic sectional curvature (the complex projective space). The found structures are defined by functions on TM, depending explicitely on the energy density (kinetic energy) on TM, only. In some cases, the obtained Kähler structures on TM have some supplimentary properties: are locally symmetric, have constant holomorphic sectional curvature etc.

In [13] we have been interested in finding a Kähler Einstein structure on the tangent bundle of a space form. We have changed the metric G on the tangent bundle (so that it is no longer obtained from a Lagrangian) by using an M-tensor field on TM obtained

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in the following way. Denote by $\tau : TM \longrightarrow M$ the canonical projection of the tangent bundle on the base manifold. Let y be the current tangent vector of TM, denote by

$$t = \frac{1}{2}g(y,y)$$

the energy density in y and let $g_y \in T^*M$ be the cotangent vector obtained from y by the "musical" isomorphism between TM and T^*M defined by g (the "lowering" of indices). Then we may consider the following symmetric M-tensor field of type $(0,2)$ on TM:

$$\tilde{G} = u(t)\ g\ +\ v(t)\ g_y \otimes g_y\ ,$$

where g is thought of as an M-tensor field on TM ([7]) and $u, v : [0,\infty) \longrightarrow \mathbf{R}$ are smooth real valued functions depending on t only. Of course, it is assumed that the values of u and v are positive and, later on, we shall study the condititions under which this property of u, v is fulfilled. If we try to find the expressions of the functions u, v in order to obtain an Einstein metric on TM defined by using the M-tensor field $\tilde{G}$, it is convenient to use an almost complex structure J on TM, related to the considered metric, such that we shall obtain, in fact, a Kähler Einstein structure on TM in the case where (M,g) has constant (negative) sectional curvature. As a matter of fact, we shall obtain the existence of a Kähler Einstein structure even in the case where (M,g) has positive constant sectional curvature, but only in a tube around the zero section in TM (Theorem 4). The surprising fact was that we have obtained on TM a structure of Kähler manifold with constant holomorphic sectional curvature (Theorem 5).

However, during the computations, we had to exclude two important cases which appeared, in a certain sense, as singular cases. One of them is the case where the metric is obtained from a Lagrangian depending on the density energy (in this case v is the derivative of u). The second special case is the case where u =constant. We have studied, separately, the case where $u = 1$, obtaining $v = -c$, where c is the constant sectional curvature of the Riemannian manifold (M,g). In this case the found Kähler Einstein structure is locally symmetric.

Remark N.Papaghiuc ([17]) observed that the case $v = 0$ must be treated as a separate (singular) case since $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ from the formulas (17), (18), (19) are no longer defined when $v = 0$. He studied separately this case and obtained a Kähler structure on the tangent bundle of a space form of constant positive sectional curvature which is not locally symmetric, is not Kähler Einstein and has not constant holomorphic sectional curvature.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart, but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always$\{1, ..., n\}$ (see [5], [4], [14], [15]). We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

Acknowledgement. When we tried to compute the components of the curvature tensor field of the considered Riemannian metric, the expressions became quite

complicated. Then we have used the Mathematica package RICCI for doing tensor calculations, elaborated by John D.Lee. Thus, some complicated expressions are not written down in this paper but it is indicated how they can be obtained by using RICCI. The author is grateful to Eng. Victor Fecioru from Brisbane, Australia, for helping him with some advices, software and hardware, necessary in the computations made by using RICCI.

1 An almost Kähler structure on the tangent bundle

Let (M, g) be a smooth $n-$dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n-$dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, ..., x^n)$ on M induces a local chart

$$(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, ...x^n,\ y^1, ..., y^n)$$

on TM where the local coordinates $x^i, y^i;\ i = 1, ..., n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau;\ i = 1, ..., n$ on TM are the local coordinates in the local chart (U, φ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n local coordinates $y^i; i = 1, ..., n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart (U, φ).

This special structure of TM allows us to introduce the notion of M- tensor field on it (see [7]).

The tangent bundle TM of a Riemannian manifold (M, g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on TM defined by g (see [18], [3]) and the complete lift type pseudo-Riemannian metric defined by g (see [19], [20], [10], [11]). Recall that the Levi Civita connection of g defines a direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \text{Ker}\ \tau_*$ and the horizontal distribution HTM defined by the Levi Civita connection $\dot{\nabla}$ of g. The vector fields $(\frac{\partial}{\partial y^1},\ ...,\frac{\partial}{\partial y^n})$ define a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, ...,\ \frac{\delta}{\delta x^n})$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\ \frac{\partial}{\partial y^h}; \quad \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g.

The distributions VTM and HTM are isomorphic each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kählerian manifold on TM (see [3]). Consider now the energy density (kinetic energy or "forza viva", according to the terminology used by Levi Civita):

$$t = \frac{1}{2} g_{ik}(x) y^i y^k \tag{2}$$

defined on TM by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U, φ). Let $u, v : [0, \infty) \longrightarrow \mathbf{R}$ be two real smooth functions. We shall assume that u, v have positive values and, later on, we shall study the conditions under which this property of u and v is fulfilled. Then we may consider the following symmetric M-tensor field of type (0,2) on TM, defined by the components (see [16], [12]:

$$G_{ij} = u(t) g_{ij} + v(t) g_{0i}\ g_{0j}\ , \tag{3}$$

where $g_{0i} = g_{hi} y^h$. The matrix (G_{ij}) is symmetric and positive and has the inverse with the entries

$$H^{kl} = \frac{1}{u} g^{kl} + w y^k y^l\ , \tag{4}$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}) and:

$$w = w(t) = -\frac{v}{u(u + 2tv)}. \tag{5}$$

The components $H^{kl}(x, y)$ define a symmetric M-tensor field of type $(2, 0)$ on TM. We shall use also the components $H_{ij}(x, y)$ of a symmetric M-tensor field of type $(0, 2)$ obtained from the components H^{kl} by "lowering" the indices:

$$H_{ij} = g_{ik} H^{kl} g_{lj} = \frac{1}{u} g_{ij} + w\ g_{0i} g_{0j}\ . \tag{6}$$

Throughout this paper we shall use also the following M-tensor fields on TM:

$$\begin{cases} G^{kl} = g^{ki} G_{ij} g^{jl} = u g^{kl} + v y^k y^l\ , \\ \\ G^i_k = G^{ih} g_{hk} = G_{kh} g^{hi} = u \delta^i_k + v\ y^i g_{0k}\ , \\ \\ H^i_k = H^{ih} g_{hk} = H_{kh} g^{hi} = \frac{1}{u} \delta^i_k + w\ y^i g_{0k}\ . \end{cases} \tag{7}$$

Remark that the matrix (H^i_k) is the inverse of the matrix (G^i_k).

The following Riemannian metric may be considered on TM:

$$G = G_{ij} dx^i dx^j + H_{ij} \dot{\nabla} y^i \dot{\nabla} y^j\ , \tag{8}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{j0} dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_{ij}, \quad G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = H_{ij}, \quad G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = \ G(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = 0.$$

Remark that HTM, VTM are orthogonal each other with respect to G but the Riemannian metrics induced from G on HTM, VTM are not the same, so the considered metric G on TM is no longer a metric of Sasaki type. Remark also that the system of 1-forms $(dx^1, ..., dx^n, \dot{\nabla}y^1, ..., \dot{\nabla}y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition (1).

An almost complex structure J may be defined on TM by:

$$J\frac{\delta}{\delta x^i} = G_i^k \frac{\partial}{\partial y^k}; \qquad J\frac{\partial}{\partial y^i} = -H_i^k \frac{\delta}{\delta x^k} \ . \tag{9}$$

Then we obtain:

Theorem 1 *(TM, G, J) is an almost Kählerian manifold.*

The associated 2-form Ω is given by

$$\Omega = g_{ij}\dot{\nabla}y^i \wedge dx^j \tag{10}$$

and it is closed since it does coincide with the 2-form associated to the Sasaki metric on TM (see [3]).

2 A Kähler structure on TM

In this section we shall study the integrability of the almost complex structure defined by J on TM. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i};\ i = 1, ..., n$:

$$[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}] = 0; \quad [\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}] = -\Gamma_{ij}^h \frac{\partial}{\partial y^h}; \quad [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = -R_{0ij}^h \frac{\partial}{\partial y^h} \ , \tag{11}$$

where $R_{0ij}^h = R_{kij}^h y^k$ and R_{kij}^h are the local coordinate components of the curvature tensor field of $\dot{\nabla}$ on M.

Theorem 2 *The almost complex structure J on TM is integrable if (M, g) has constant sectional curvature c and the function v is given by:*

$$v = \frac{c - uu'}{2tu' - u}. \tag{12}$$

Of course we have to study the conditions under which v has positive values. From the condition

$$N_J = 0$$

it follows that the curvature tensor field of $\dot{\nabla}$ must have the expression

$$R_{hij}^k = c(\delta_i^k g_{hj} - \delta_j^k g_{hi}) \ , \tag{13}$$

where c is a constant and then we obtain the expression (12) of v. Next, we obtain easily the expression of the function w:

$$w = w(t) = \frac{uu' - c}{u(2tc - u^2)} \ . \tag{14}$$

3 A Kähler Einstein structure on TM

In this section we shall study the property of (TM, G) to be Einstein. We shall find the expression of the Levi Civita connection ∇ of the metric G on TM, then we shall find the expression of the curvature tensor field of ∇. Then, computing the corresponding traces we shall find the components of the Ricci tensor field of ∇. Asking for the Ricci tensor field to be proportional with the metric G we find a second order ordinary differential equation which must be fulfilled by the function u. Fortunately, we have been able to find the general solution of this differential equation. For a special value of one of the integration constants we obtained the property of G to be Einstein. At the same time we have been able to study the conditions under which the functions u, v, w have positive values.

Theorem 3 *([16]) The Levi Civita connection ∇ of G has the following expression in the local adapted frame $(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n})$:*

$$(15)\qquad \begin{cases} \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij} \frac{\partial}{\partial y^h}\ ,\ \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h} + P^h_{ji} \frac{\delta}{\delta x^h}\ , \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij} \frac{\delta}{\delta x^h}\ ,\ \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + S^h_{ij} \frac{\partial}{\partial y^h}, \end{cases}$$

where the M-tensor fields $P^h_{ij},\ Q^h_{ij},\ S^h_{ij}$ are given by:

$$(16)\qquad \begin{cases} P^h_{ij} = \frac{1}{2}(\frac{\partial G_{jk}}{\partial y^i} + H_{il} R^l_{0jk}) H^{kh}\ , \\ Q^h_{ij} = \frac{1}{2} G^{hk} (\frac{\partial H_{jk}}{\partial y^i} + \frac{\partial H_{ik}}{\partial y^j} - \frac{\partial H_{ij}}{\partial y^k})\ , \\ S^h_{ij} = \frac{1}{2}(-R^h_{0ij} - \frac{\partial G_{ij}}{\partial y^k} G^{kh})\ . \end{cases}$$

Taking into account the expressions (3),(6) of G_{ij} and H_{ij} and by using the formulas (12),(14),(5) we may obtain the following expressions:

$$(17)\qquad \begin{cases} P^h_{ij} = \frac{u'}{2u} \delta^h_j g_{0i} + \frac{uv-c}{2u^2} \delta^h_i g_{0j} - \\ -\frac{(c+uv)w}{2v} g_{ij} y^h + \frac{vw(uv-c)+uw(u'v-uv')}{2uv} g_{0i} g_{0j} y^h\ , \end{cases}$$

$$(18)\qquad \begin{cases} Q^h_{ij} = -\frac{u'}{2u}(\delta^h_j g_{0i} + \delta^h_i g_{0j}) - \\ -\frac{v(u'+2u^2w)}{2u^3w} g_{ij} y^h - \frac{v(2u'w+uw')}{2u^2w} g_{0i} g_{0j} y^h\ , \end{cases}$$

$$(19)\qquad \begin{cases} S^h_{ij} = \frac{c-uv}{2} \delta^h_j g_{0i} - \frac{c+uv}{2} \delta^h_i g_{0j} + \\ +\frac{u'v}{2uw} g_{ij} y^h + \frac{v(v'-2uvw)}{2uw} g_{0i} g_{0j} y^h\ . \end{cases}$$

Remark that in the obtained formulas we have used the formula (5) in order to replace the energy density t, such that it is not involved explicitly. Of course, we can replace v, v', w, w' as functions of u, u', u'' and t but we obtain much more complicated expressions for $P_{ij}^h, Q_{ij}^h, S_{ij}^h$.

The curvature tensor field K of the connection ∇ is defined by the well known formula:

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ : \quad X, Y, Z \in \Gamma(TM) \ .$$

By using the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$; $i = 1, \dots, n$ we obtain, after a standard straightforward computation:

$$K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = XXX_{kij}^h \frac{\delta}{\delta x^j} \ ; \ K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = XXY_{kij}^h \frac{\partial}{\partial y^h} \ ;$$

$$K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} = YYX_{kij}^h \frac{\delta}{\delta x^h} \ ; \ K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = YYY_{kij}^h \frac{\partial}{\partial y^h} \ ;$$

$$K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = YXX_{kij}^h \frac{\partial}{\partial y^h} \ ; \ K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = YXY_{kij}^h \frac{\delta}{\delta x^h}.$$

where

$$(20) \qquad \begin{cases} XXX_{kij}^h = R_{kij}^h + P_{lk}^h R_{0ij}^l + P_{li}^h S_{jk}^l - P_{lj}^h S_{ik}^l \ , \\ XXY_{kij}^h = R_{kij}^h + Q_{lk}^h R_{0ij}^l + S_{il}^h P_{kj}^l - S_{jl}^h P_{ki}^l \ , \\ YYX_{kij}^h = \frac{\partial}{\partial y^i} P_{jk}^h - \frac{\partial}{\partial y^j} P_{ik}^h + P_{il}^h P_{jk}^l - P_{jl}^h P_{ik}^l \ , \\ YYY_{kij}^h = \frac{\partial}{\partial y^i} Q_{jk}^h - \frac{\partial}{\partial y^j} Q_{ik}^h + Q_{il}^h Q_{jk}^l - Q_{jl}^h Q_{ik}^l \ , \\ YXX_{kij}^h = \frac{\partial}{\partial y^i} S_{jk}^h + Q_{il}^h S_{jk}^l - S_{jl}^h P_{ik}^l \ , \\ YXY_{kij}^h = \frac{\partial}{\partial y^i} P_{kj}^h + P_{il}^h P_{kj}^l - P_{lj}^h Q_{ik}^l. \end{cases}$$

Remark that, at a first step, the formulas for the expressions of K contained also some other terms involving the Christoffel symbols Γ_{ij}^k. However, after some standard computations, we have been able to show that those other terms are zero.

Now we have to replace the expressions (17), (18), (19) of the M-tensor fields $P_{ij}^h, Q_{ij}^h, S_{ij}^h$ and the expressions (12),(14),(5) of the functions v, w and of their derivatives in order to obtain the components from (20) as functions of u, u', u'', $u^{(3)}$ only. The obtained expressions are quite complicated and, at this stage, we decided to use the Mathematica package RICCI in order to do the necessary tensor calculations. It has been useful to consider $c, t, u, v, w, u', v', w'$ (denoted as up, vp, wp respectively),u'', v'', w'' (denoted as us, vs, ws respectively) and $u^{(3)}$ (denoted as ut) as constants, the tangent vector y as a first order tensor, the components G_{ij}, H_{ij} as second order tensors and so on, on the tangent bundle to the Riemannian manifold M, the associated indices being h, i, j, k, l, r, s. It was not convenient to think u, v, w as functions of t since RICCI did not make some useful factorizations after the command TensorSimplify.

The components of the Ricci tensor field of the connection ∇, defined as $Ric(Y,Z) =$ trace $(X \longrightarrow K(X,Y)Z)$; $X, Y, Z \in \Gamma(TM)$ are obtained as follows:

$$Ric(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) = RicXX_{jk} = XXX^h_{khj} + YXX^h_{khj} ,$$

$$Ric(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}) = RicYY_{jk} = YYY^h_{khj} - YXY^h_{kjh} ,$$

$$Ric(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}) = Ric(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}) = 0 .$$

The expressions of $RicXX_{jk}, RicYY_{jk}$ are quite complicated. In order to present a summary description of these expressions we introduce the function:

$$a = n(u-2tu')(2cu-2ctu'-u^2u') + 2(2ct-u^2)(tuu''+uu'-tu'^2) . \tag{21}$$

Then:

$$RicXX_{jk} = \frac{a}{2(u-2tu')^2} g_{jk} + \alpha(t,u,u',u'',u^{(3)}) g_{0j}g_{0k}$$

$$RicYY_{jk} = \frac{a}{2u^2(u-2tu')^2} g_{jk} + \beta(t,u,u',u'',u^{(3)}) g_{0j}g_{0k},$$

where α, β are rational expressions in $t, u, u', u'', u^{(3)}$.

To study the conditions under which (TM, G) is Einstein, we consider the differences:

$$DiffXX_{jk} = RicXX_{jk} - \frac{a}{2u(u-2tu')^2} G_{jk} ,$$

$$DiffYY_{jk} = RicYY_{jk} - \frac{a}{2u(u-2tu')^2} H_{jk} .$$

Then we obtain:

$$DiffXX_{jk} = \frac{1}{2u^2(u-2tu')^4}[n(u^2-2ct)(2tu'-u)(u^2u''-2tu'^3+2uu'^2)-$$
$$-8c^2tu^3u' + 4cu^5u' + 16c^2t^2u^2u'^2 + 4ctu^4u'^2 - 6u^6u'^2 - 24c^2t^3uu'^3 - 8ct^2u^3u'^3 +$$
$$+10tu^5u'^3 + 16c^2t^4u'^4 - 4t^2u^4u'^4 - 24c^2t^2u^3u'' + 20ctu^5u'' - 4u^7u'' +$$
$$+16c^2t^3u^2u'u'' - 4tu^6u'u'' - 16ct^3u^3u'^2u'' + 8t^2u^5u'^2u'' - 32c^2t^4u^2u''^2 + 32ct^3u^4u''^2 -$$
$$-8t^2u^6u''^2 - 8c^2t^3u^3u^{(3)} + 8ct^2u^5u^{(3)} - 2tu^7u^{(3)} + 16c^2t^4u^2u'u^{(3)} - 16ct^3u^4u'u^{(3)} +$$
$$+4t^2u^6u'u^{(3)}]g_{0j}g_{0k} ,$$

$$DiffYY_{jk} = \frac{1}{2u^2(u^2-2ct)(u-2tu')^2}[n(u^2-2ct)(2tu'-u)(u^2u''-2tu'^3+2uu'^2)+$$

$$+4cu^3u' - 8ctu^2u'^2 - 6u^4u'^2 + 12ct^2uu'^3 + 10tu^3u'^3 - 8ct^3u'^4 - 4t^2u^2u'^4 + 12ctu^3u'' -$$
$$-4u^5u'' - 8ct^2u^2u'u'' - 4tu^4u'u'' + 8t^2u^3u'^2u'' + 16ct^3u^2u''^2 - 8t^2u^4u''^2 + 4ct^2u^3u^{(3)} -$$
$$-2tu^5u^{(3)} - 8ct^3u^2u'u^{(3)} + 4t^2u^2u'u^{(3)}]g_{0j}g_{0k} .$$

Our task is to find a positive function $u(t)$ such that $DiffXX_{jk} = DiffYY_{jk} = 0$. It was hopeless to try to find directly a general common solution of the system of third order ordinary differential equations obtained by imposing the conditions $DiffXX_{jk} = DiffYY_{jk} = 0$.

If we try to find a function u for which $DiffXX_{jk} = DiffYY_{jk} = 0$ and which does not depend on the dimension n of M, then we have to find the positive solutions of the second order differential equation:

$$u^2u'' - 2tu'^3 + 2uu'^2 = 0 . \tag{22}$$

We succeeded to find the general solution of the equation (22), then we studied the conditions under which the symmetric matrix (G_{jk}) is positive, obtaining the following cases:

1. $c < 0$; $A > 0$ and we have:

$$u = A + \sqrt{A^2 - 2ct} > 0 \ , \ v = \frac{1}{2t}(A - \frac{4ct}{A} - \sqrt{A^2 - 2ct}) > 0 \ ; \ t \geq 0 . \tag{23}$$

2. $c > 0$; $A > 0$ and we have:

$$u = A \pm \sqrt{A^2 - 2ct} \ ; \ v = \frac{1}{2t}(A - \frac{4ct}{A} \mp \sqrt{A^2 - 2ct}) \ ; \ 0 \leq t < \frac{A^2}{2c} . \tag{24}$$

Remark that we may have $u > 0$, $v \leq 0$ but the matrix (G_{jk}) is still positive since, for $z, y \in TM$; $\tau(z) = \tau(y) = x \in M$, we have:

$$G_{(x,y)}(z,z) = u\|z\|^2 + v < y, z >^2 \geq \|z\|^2(u + 2tv),$$

where $\|z\|^2 = g_x(z,z)$, $2t = g_x(y,y) = \|y\|^2$. Hence it is enough to have $u + 2tv > 0$.

Hence, we may state our main result

Theorem 4 *1. Assume that (M,g) has constant negative sectional curvature c and let $A > 0$. Then (TM, G, J), with u, v given by (23) is a Kähler Einstein manifold.*

2. Assume that (M,g) has constant positive sectional curvature c and let $A > 0$. Then the tube around the zero section in TM, defined by the condition:

$$g_{jk}y^jy^k = 2t < \frac{A^2}{c}$$

has a structure of Kähler Einstein manifold, if the functions u, v are given by (24).

Remark The case $c = 0$ is not considered since (M,g) is flat and we obtain just the flat Kähler structure on TM defined in [3].

4 The holomorphic sectional curvature of (TM, G, J)

In this section we shall obtain the components of the curvature tensor field K of ∇ in the case where u, v are given by (23), with $c < 0, A > 0$. Similar computations may be done in the remaining cases. Replacing in the formulas (20) P, Q, S with their expressions (17), (18), (19), where $u, u', u'', u^{(3)}, v, v', v'', w, w', w''$ are computed by using (23), we obtain the following quite simple expressions of the components (of course, we have used RICCI to do the corresponding tensor calculations):

$$(25)\qquad \begin{cases} XXX^h_{kij} = \frac{c}{2A}(\delta^h_i G_j k - \delta^h_j G_{ik}) \ ; \ XXY^h_{kij} = \frac{c}{2A}(g_{jk} G^h_i - g_{ik} G^h_j) \ ; \\ YYX^h_{kij} = \frac{c}{2A}(g_{jk} H^h_i - g_{ik} H^h_j) \ ; \ YYY^h_{kij} = \frac{c}{2A}(\delta^h_i H_{jk} - \delta^h_j H_{ik}) \ ; \\ YXX^h_{kij} = \frac{c}{2A}(\delta^h_i G j k + g_{ik} G^h_j + 2 g_{ij} G^h_k) \ ; \\ YXY^h_{kij} = -\frac{c}{2A}(\delta^h_j H_{ik} + g_{jk} H^h_i + 2 g_{ij} H^h_k) \ . \end{cases}$$

Recall that a Kähler manifold (M, g, J) has holomorphic constant sectional curvature k if its curvature tensor field R is given by:

$$(26)\qquad R(X,Y)Z = \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + $$
$$+2g(X,JY)JZ\} \ ; \ X, Y, Z \in \chi(M) \ .$$

Comparing the expressions (25) of the components of K with those obtained from (26), when we take $M \to TM$, $g \to G$ and for the vector fields X, Y, Z involved in (28) we take the elements of the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$; $i = 1, \ldots, n$, we obtain a quite interesting result:

Theorem 5 *If $A > 0, c < 0$ and (TM, G, J) has the Kähler Einstein structure defined by the expression (23) of u and v, then (TM, G, J) is a complex space form with negative constant holomorphic sectional curvature $\frac{2c}{A}$.*

Similar results are obtained in the cases where u, v are given by (24).

The components of the Ricci tensor field Ric of ∇ are:

$$RicXX_{jk} = \frac{(n+1)c}{A} G_{jk} \ ; \ RicYY_{jk} = \frac{(n+1)c}{A} H_{jk} \ ; \ RicXY_{jk} = RicYX_{jk} = 0,$$

thus:

$$Ric = \frac{(n+1)c}{A} G \ .$$

5 The case where $u = 1$

In this section we shall study separately the case where u =constant. In fact, we shall take $u = 1$.

Theorem 6 *If $u = 1$, then the almost complex structure J on TM is integrable if and only if (M, g) has constant sectional curvature c and we have*

$$v = -c. \tag{27}$$

If $c < 0$ we obtain a Kähler Einstein structure on whole TM. In the case where the constant c is positive we have a Kähler structure in the tube around the zero section in TM defined by

$$t < \frac{1}{2c}. \tag{28}$$

Then the components G_{ij}, H_{ij} of the M-tensor fields on TM, defining the Riemannian metric G are given by:

$$\begin{cases} G_{ij} = g_{ij} - c\, g_{0i}g_{0j}\ , \\ H_{ij} = g_{ij} + \frac{c}{1-2ct}\, g_{0i}g_{0j}\ . \end{cases} \tag{29}$$

From the expression of G_{ij} it follows that the symmetric matrix (G_{ij}) is positive if $c < 0$ and from the expression of H_{ij} it follows that we have a Kähler structure even when $c > 0$ but only in the region of TM where $g_{ij}y^iy^j = 2t < \frac{1}{c}$.

The Levi Civita connection of the metric G may be obtained easily.

Theorem 7 *The Levi Civita connection ∇ of the Kähler manifold (TM, G, J) is given by the formulas:*

$$\begin{cases} \nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = Q^h_{ij}\frac{\partial}{\partial y^h}\ ,\ \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \Gamma^h_{ij}\frac{\partial}{\partial y^h} + P^h_{ji}\frac{\delta}{\delta x^h}\ , \\ \nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = P^h_{ij}\frac{\delta}{\delta x^h}\ ,\ \nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \Gamma^h_{ij}\frac{\delta}{\delta x^h} + S^h_{ij}\frac{\partial}{\partial y^h}, \end{cases} \tag{30}$$

where the M-tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by:

$$\begin{cases} P^h_{ij} = -c\delta^h_i g_{0j} - \frac{c^2}{1-2ct}g_{0i}g_{0j}y^h = -cg_{0j}H^h_i\ , \\ Q^h_{ij} = cg_{ij}y^h + \frac{c^2}{1-2ct}g_{0i}g_{0j}y^h = cy^hH_{ij}\ , \\ S^h_{ij} = c\delta^h_j g_{0i} - c^2 g_{0i}g_{0j}y^h = cg_{0i}G^h_j\ . \end{cases} \tag{31}$$

Denote by K the curvature tensor field of the connection ∇. Then we obtain by a straightforward computation:

$$\begin{cases} K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = c(\delta^h_i G_{jk} - \delta^h_j G_{ik})\frac{\delta}{\delta x^h}\ , \\ K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = c(g_{jk}G^h_i - g_{ik}G^h_j)\frac{\partial}{\partial y^h}\ , \\ K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} = c(g_{jk}H^h_i - g_{ik}H^h_j)\frac{\delta}{\delta x^h}\ , \\ K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = c(\delta^h_i H_{jk} - \delta^h_j H_{ik})\frac{\partial}{\partial y^h}\ , \\ K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = cg_{ij}G^h_k\frac{\partial}{\partial y^h}\ ,\ K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = -cg_{ij}H^h_k\frac{\delta}{\delta x^h}. \end{cases} \tag{32}$$

Finally, we obtain the Ricci tensor field of ∇, defined as $Ric(Y,Z) = \mathrm{trace}(X \longrightarrow K(X,Y)Z)$; $X,Y,Z \in \Gamma(TM)$. It follows:

$$(33)\qquad \begin{cases} Ric(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = c\ n\ G_{jk}\ , \\ Ric(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = c\ n\ H_{jk}\ , \\ Ric(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = 0. \end{cases}$$

It follows that $Ric = c\ n\ G$.

By using the the expression (32) of K, we computed its the covariant derivatives with respect to the connection ∇, by using the vector fields $\frac{\delta}{\delta x^h}$ and $\frac{\partial}{\partial y^h}$, and and we obtained that in all twelve cases the result is zero. Thus we may state:

Theorem 8 *If (M,g) has constant negative sectional curvature and the components of the metric G are given by (13) then (TM,G) is a locally symmetric space. If (M,g) has constant positive sectional curvature and the components of the metric G are given by (13) then the tube around the zero section in TM defined by the condition (12), endowed with the metric G, is a locally symmetric Riemannian manifold.*

Remark It is well known that an irreducible locally symmetric Riemannian manifold is automatically Einstein, thus our result $Ric = c\,n\,G$ is a consequence of the Theorem 8. However, it seemed to us that it is simpler to compute the traces of K in order to obtain Ric than to to compute the components of ∇K.

6 The case where $v = u'$

In this section we shall present the results obtained by the author and N.Papaghiuc in [16] in the case where $v = u'$.

First of all we obtained the following integrability condition for the almost complex structure J on TM.

Theorem 9 *The almost complex structure J on TM is integrable if and only if (M,g) has positive constant sectional curvature c and the function $u(t)$ satisfies the ordinary differential equation*

$$(34)\qquad 2t(u')^2 = c.$$

The general solution of the differential equation (34) may be obtained easily. Since we look for a solution u defined for $t > 0$, for which $u > 0$, $u' > 0$, we may take

$$(35)\qquad u' = \sqrt{\frac{c}{2t}}; \qquad u = \sqrt{2ct}\ ,$$

and:

$$w = -\frac{1}{4t\sqrt{2ct}}.$$

Remark that the functions $u,\ u', w$ are smooth only on the nonzero tangent vectors of M. Hence we obtain, in fact, a Kähler structure only on the manifold T_0M=the bundle of nonzero tangent vectors to M.

In order to obtain the expression of the Levi Civita connection ∇ on TM, determined by the metric G we shall introduce, for convenience, the following M-tensor fields on T_0M:

$$a_{ij} = g_{ij} - \frac{1}{2t}g_{0i}g_{0j}, \quad a_i^k = \delta_i^k - \frac{1}{2t}g_{0i}y^k. \tag{36}$$

Remark that we have

$$a_{i0} = a_{0i} = a_{ij}y^j = 0$$

and

$$a_{ij} = a_i^k g_{kj} = a_i^k a_{kj}.$$

Then we obtain

$$\begin{cases} G_{ij} = \sqrt{2ct}g_{ij} + \sqrt{\frac{c}{2t}}g_{0i}\ g_{0j} = \sqrt{2ct}(a_{ij} + \frac{1}{t}g_{0i}g_{0j}), \\ H_{ij} = \frac{1}{\sqrt{2ct}}g_{ij} - \frac{1}{4t\sqrt{2ct}}\ g_{0i}g_{0j} = \frac{1}{\sqrt{2ct}}(a_{ij} + \frac{1}{4t}g_{0i}g_{0j}), \\ H^{jk} = \frac{1}{\sqrt{2ct}}g^{jk} - \frac{1}{4t\sqrt{2ct}}y^jy^k; \quad G^{jk} = \sqrt{2ct}g^{jk} + \sqrt{\frac{c}{2t}}y^jy^k. \end{cases} \tag{37}$$

and:

Theorem 10 *The Levi Civita connection of the Kähler manifold (T_0M, G, J) is given by*

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} &= (-\frac{1}{4t}g_{0i}\delta_j^h - \frac{1}{4t}g_{0j}\delta_i^h + \frac{1}{8t^2}g_{0i}g_{0j}y^h)\frac{\partial}{\partial y^h} \\
\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} &= \Gamma_{ij}^h\frac{\partial}{\partial y^h} + (\frac{1}{4t}g_{ij}y^h + \frac{1}{4t}g_{0j}\delta_i^h - \frac{1}{8t^2}g_{0i}g_{0j}y^h)\frac{\delta}{\delta x^h} = \\
&= \Gamma_{ij}^h\frac{\partial}{\partial y^h} + (\frac{1}{4t}a_{ij}y^h + \frac{1}{4t}\delta_i^h g_{0j})\frac{\delta}{\delta x^h} \\
\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} &= (\frac{1}{4t}g_{ij}y^h + \frac{1}{4t}g_{0i}\delta_j^h - \frac{1}{8t^2}g_{0i}g_{0j}y^h)\frac{\delta}{\delta x^h} = \\
&= (\frac{1}{4t}a_{ij}y^h + \frac{1}{4t}\delta_j^h g_{0i})\frac{\delta}{\delta x^h} \\
\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} &= \Gamma_{ij}^h\frac{\delta}{\delta x^h} - c(g_{ij}y^h + \delta_i^h g_{0j})\frac{\partial}{\partial y^h}.
\end{aligned}$$

Then the expression of the operator J is given by:

$$J\frac{\delta}{\delta x^i} = (\sqrt{2ct}\delta_i^k + \sqrt{\frac{c}{2t}}g_{0i}y^k)\frac{\partial}{\partial y^k}$$

$$J\frac{\partial}{\partial y^i} = (-\frac{1}{\sqrt{2ct}}\delta_i^k + \frac{1}{4t\sqrt{2ct}}g_{0i}y^k)\frac{\delta}{\delta x^k},$$

and it can be checked easily that $\nabla J = 0$.

The expression of the curvature tensor field K of the Levi Civita connection ∇ on T_0M is obtained easily by a straightforward computation.

Theorem 11 *The local coordinate expression of the curvature tensor field K of the Kähler manifold (T_0M, G, J) is given in the adapted local frame $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i})$ by:*

$$\left\{\begin{array}{ll} K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = \frac{1}{4t}K^h_{kij}\frac{\partial}{\partial y^h}; & K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} = \frac{1}{4t}K^h_{kij}\frac{\delta}{\delta x^h}; \\ K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = \frac{1}{4t}S^h_{kij}\frac{\delta}{\delta x^h}; & K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = -\frac{c}{2}S^h_{kij}\frac{\partial}{\partial y^h}; \\ K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = \frac{c}{2}K^h_{kij}\frac{\partial}{\partial y^h}; & K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = \frac{c}{2}K^h_{kij}\frac{\delta}{\delta x^h}, \end{array}\right. \tag{38}$$

where we have denoted

$$K^h_{kij} = \frac{1}{c}\{R^h_{kij} - \frac{1}{2t}g_{0k}R^h_{0ij} + \frac{1}{2t}g_{lk}R^l_{0ij}y^h\} = a^h_i a_{jk} - a^h_j a_{ik} \tag{39}$$

and

$$\begin{gathered} S^h_{kij} = g_{ik}\delta^h_j + g_{jk}\delta^h_i - \\ -\frac{1}{2t}[g_{0i}g_{jk}y^h + g_{0j}g_{ik}y^h + g_{0i}g_{0k}\delta^h_j + g_{0j}g_{0k}\delta^h_i] + \frac{1}{2t^2}g_{0i}g_{0j}g_{0k}y^h = \\ = a^h_i a_{jk} + a^h_j a_{ik}\ . \end{gathered} \tag{40}$$

From the above formulas, we get by a straightforward computation that the local coordinate expression of the Ricci tensor $S(Y, Z) = \mathrm{trace}(X \longrightarrow K(X, Y)Z)$ in the local frame adapted to the direct sum decomposition (1) is given by

$$\left\{\begin{array}{l} S(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = -\frac{1}{2t}\ [g_{ij} - \frac{1}{2t}g_{0i}g_{0j}] = -\frac{1}{2t}a_{ij}, \\ S(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = -c[g_{ij} - \frac{1}{2t}\ g_{0i}g_{0j}] = -ca_{ij}, \\ S(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = 0. \end{array}\right. \tag{41}$$

Comparing the obtained expressions with the expressions (37) of the components of G we obtain:

Proposition 1 *The Kählerian manifold (T_0M, G, J) cannot be an Einstein manifold.*

From the expression (38) of K it follows also:

Proposition 2 *The Kählerian manifold (T_0M, G, J) cannot have constant holomorphic sectional curvature.*

Finally, by computing the covariant derivative of K with respect to ∇, we obtain.

Proposition 3 *The Kählerian manifold (T_0M, G, J) cannot be locally symmetric.*

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