The Geometry of a Vector Bundle Endowen with a Cone Field

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Abstract

In this research-expository survey the definition of a regular vector bundle endowed with a cone field and some fundamental elements of the geometry of this new structure are presented. Two particular examples of this structure were studied in [13] and in [14]. This way, in [13] a new geometry of a Lorentzian manifold and in [14] a new geometry of the set of Riemannian metrics and of the group of diffeomorphisms of an arbitrary differentiable manifold are obtained.[1]

Definition of a regular bundle endowed with a cone field

A regular vector bundle *is a vector bundle* (E, p, M) *for which* M *is a real topological paracompact connected manifold without boundary,* $\dim M = n$, $\dim E = n + m$.

A cone field of (E, p, M) *is a map* K,

$$K : x \in M \to K(x) \subset E_x \subset E, \quad (E_x = p^{-1}(x)),$$

for which the following two axioms are satisfied:

A_1. $(\forall x \in M)$, $K(x)$ *is a convex pointed closed with interior points cone of* E_x.
[[*Cone* $K(x) \Leftrightarrow$ *a subset of* E_x *such that the relation* $X \in K(x)$ *implies* $\lambda X \in K(x)$ $(\forall \lambda \in \mathbf{R},\ \lambda \geq 0)$. *Pointed* $\Leftrightarrow$ *if* $X \in K(x)$ *and* $-X \in K(x)$ *then* $X = 0$.]]

A_2. *The sets* $\bigcup_{x\in M} \operatorname{Int} K(x)$ *and* $\bigcup_{x \in M} (E_x - K(x))$ *are open subsets of the topological manifold* E.

A regular vector bundle (E, p, M) endowed with **a cone field** K will be denoted by $[(E, p, M); K]$.

A regular vector bundle (E, p, M) can have a cone field if and only if it has a continuous global section of non-zero vectors.

The research of a structure $[(E, p, M); K]$ **was suggested by**

- the theory of *Krein spaces* (B, K) [[B is a Banach space and K is a convex pointed closed with interior points cone of B]],
- the theory of positive operators on *Krein spaces* [[for a *Krein-space* (B, K), a *positive operator* is a *continuous map* $\phi : B \to B$ such that $\phi(K) \subset K$]].

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Two monographs about these topics were published by a group of mathematicians from Voronesz, Russia ([5] Mark Krasnosel'skij, 1964; [6] Mark A. Krasnosel'skij, Je. A. Lifshits, A. V. Sobolev, 1989).
The research of a structure $[(E,p,M);K]$ **was recommended and justified** by
- the results obtained in the study of *global sections of* (E,p,M) [7–11] and in the study of *positive operators* [1–4, 12],
- the existence till now of two important particular examples, namely
the tangent vector bundle of a time-space (i.e. of a time-oriented Lorentzian manifold) [13],
the vector bundle of 2*-times covariant symmetric tensors tangent to a smooth manifold*, structure denoted by $(ST_2(M),p,M)$ [14].
From 1988, the geometry of some structures associated to an arbitrary regular vector bundle endowed with a cone field has been studied by Dan I. Papuc and Liana David in 12 papers (see the **References**).
The first paper where a particular cone field on a certain regular vector bundle associated to a differentiable manifold was used, seems to be [15] (Denis Sullivan, 1976).

The geometry of a regular vector bundle endowed with a cone field

1 The geometry of a local fibre of a structure $[(E,p,M);K]$

The pair $(E_x;K(x))$, $(\forall x\in M)$, is *a Krein space.* Hence it follows:
a). There is **an ordering relation** on E_x:

$$X\le Y \overset{def}{\Leftrightarrow} Y-X\in K(x),\quad (X,Y\in E_x).$$

$(E_x;\le)$ is *an ordered topological vector space directed on both sides.*
b). For every $Z\in \operatorname{Int} K(x)$, E_x is **Z-measurable**, i.e.

$$\forall X\in E_x,\quad \exists\lambda\in\mathbf{R},\ \lambda\ge 0\mid -\lambda Z\le X\le \lambda Z.$$

Consequently E_x has **a norm** determined by $K(x)$ and a fixed $Z\in\operatorname{Int}K(x)$:

$$|\cdot|_Z : X\in E_x\to |X|_Z\in\mathbf{R},$$
$$|X|_Z\overset{def}{=}\min\{\lambda\mid \lambda\in\mathbf{R},\ \lambda\ge 0,\ -\lambda Z\le X\le\lambda Z\}.$$

This *norm is monotone* $(0\le X\le Y\Rightarrow 0\le |X|_Z\le |Y|_Z)$.
c). The **open-ball** of E_x in the *norm* $|\cdot|_Z$ and the **open ordered** (in $\le$) **interval**, both determined by the same $X_0\in E_x$, $\varepsilon\in\mathbf{R}$ $(\varepsilon>0)$ and $Z\in\operatorname{Int}K(x)$, coincide, i.e.

$$B(X_0,Z,\varepsilon)=(X_0-\varepsilon Z, X_0+\varepsilon Z),$$

where

$$B(X_0,Z,\varepsilon)\overset{def}{=}\{X\mid X\in E_x,\ |X-X_0|_Z<\varepsilon\},$$

$$(X_0 - \varepsilon Z, X_0 - \varepsilon Z) \stackrel{def}{=} \{X \mid X \in E_x,\ \exists \varepsilon_1,\ 0 \leq \varepsilon_1 < \varepsilon,$$
$$X_0 - \varepsilon_1 Z \leq X \leq X_0 + \varepsilon_1 Z\}.$$

The three *topologies* of E_x which are determined, the first by the *structure* (E, p, M), the second by the *norm* $|\cdot|_Z$ and the third by *open ordered* (in $\geq$, determined by Z) *intervals*, coincide.
d). In the projective space of directions determined by non-zero vectors from E_x, for two arbitrary directions, $\lambda_a Z_a$, $a = 1, 2$, $Z_a \in \operatorname{Int} K(x)$ **a distance** is defined by the absolute value of Napierian logarithm of double ratio of directions

$$\rho(\lambda_1 Z_1, \lambda_2 Z_2) \stackrel{def}{=} |\ln[\mu_1 U_1, \mu_2 U_2; \lambda_1 Z_1, \lambda_2 Z_2]|$$

where $U_a \in \operatorname{Fr} K(x)$, and Z_a, U_a, $(a = 1, 2)$ belong to the same 2-dimensional subspace of E_x.
The geometry of the manifold of directions from $\operatorname{Int} K(x)$ *endowed with the above distance is analogous to the Lobachewski-Bolyai geometry.*

2 The fundamental theorem of a structure $[(E, p, M); K]$

a). In order to study a structure $[(E, p, M); K]$ a main tool is the function

$$v : (Z, X) \in \bigcup_{x \in M} (\operatorname{Int} K(x) \times E_x) \to v(Z, X) \in \mathbf{R}^2\ ,$$

where $v(Z, X) = (\alpha(Z, X), \beta(Z, X))$ and

$$\alpha(Z, X) \stackrel{def}{=} \min\{\lambda \mid \lambda \in \mathbf{R},\ X \leq \lambda Z\},$$

$$\beta(Z, X) \stackrel{def}{=} \max\{\lambda \mid \lambda \in \mathbf{R},\ \lambda Z \leq X\}.$$

The function v has the following fundamental properties:

$$\forall (Z, X), (Z, Y) \in \bigcup_{x \in M} (\operatorname{Int} K(x) \times E_x)\ ,$$

I. $\forall \lambda \in \mathbf{R},\ \lambda > 0 \mid v(Z, \lambda X) = \lambda . v(Z, X),\ \ v(\lambda Z, X) = (1/\lambda) . v(Z, X)$.
II. $\beta(Z, X) + \beta(Z, Y) \leq \beta(Z, X + Y) \leq \alpha(Z, X + Y) \leq \alpha(Z, X) + \alpha(Z, Y)$.
III. $v(Z, Z) = (1, 1)$, $v(Z, X) = (0, 0) \Rightarrow X = 0$.
IV. $\forall \lambda \in \mathbf{R} \mid \alpha(Z, \lambda Z - X) = \lambda - \beta(Z, X)$, $\beta(Z, \lambda Z - X) = \lambda - \alpha(Z, X)$.
The map v is a continuous map.
The next theorem proves that a structure $[(E, p, M); K]$ can be determined by the following three "classical" elements:
1) *a vector bundle* (E, p, M),
2) *a continuous global sections of non-zero elements of* (E, p, M) *and*
3) *a certain function defined on* E *with values in* $\mathbf{R}^2$.

Fundamental theorem:
If *it is given*
- a regular vector bundle (E, p, M),
- a global C^0 *section* ζ *of* (E, p, M) *so that* $\zeta(x) \neq 0$ $(\forall x \in M)$,
- a global C^0 *function* $v_\zeta : X \in E \to (\alpha_\zeta(X), \beta_\zeta(X)) \in \mathbf{R}^2$ *satisfying the conditions obtained from the fundamentals properties I–IV of* v*, substituing there*

$$v(\zeta(p(X)), X) = (\alpha(\zeta(p(X)), X), \beta(\zeta(p(X)), X)) \in \mathbf{R}^2$$

by

$$v_\zeta(X) = (\alpha_\zeta(X), \beta_\zeta(X)), \quad (\forall X \in E),$$

then *these three elements* $(E, p, M), \zeta, v_\zeta$*, uniquely determine a structure* $[(E, p, M); K]$*, where* $\forall x \in M \mid K(x) = \{X \in E_x \mid \beta_\zeta(X) \geq 0\}$.
If in E_x *we consider the ordering* $X \leq Y \Leftrightarrow Y - X \in K(x)$*, then*

$$\alpha_\zeta(X) = \min\{\lambda \mid \lambda \in \mathbf{R}, X \leq \lambda\zeta(x)\},\ \beta_\zeta(X) = \max\{\lambda \mid \lambda \in \mathbf{R},\ \lambda\zeta(x) \leq X\}.$$

b). Some important relations:

$$\forall X \in E_x,\ Z, Z_1, Z_2 \in \operatorname{Int} K(x),\ U_1, U_2 \in \operatorname{Fr} K(x)\ ,$$

1. $X \in K(x) \Leftrightarrow \beta(Z, X) \geq 0$.
2. $X \leq Y \Leftrightarrow \beta(Z, Y - X) \geq 0$.
3. $|X|_Z = \max\{|\alpha(Z, Y)|,\ |\beta(Z, X)|\}$.
4. $\rho(\lambda_1 Z_1, \lambda_2 Z_2) = |\ln[\mu_1 U_1, \mu_2 U_2; \lambda_1 Z_1, \lambda_2 Z_2]| =$
$= |\ln(\beta(Z_1, Z_2)/\alpha(Z_1, Z_2))| = |\ln(\beta(Z_2, Z_1)/\alpha(Z_2, Z_1))|$.

3 The space of global sections of a structure $[(E, p, M); K]$

a). Definitions and notations.
The set of all global sections of the vector bundle (E, p, M) is

$$\Sigma \stackrel{def}{=} \{\sigma \mid \sigma : M \to E,\ p \circ \sigma = \mathrm{id}_M,\ \sigma \text{ is } C^0\}.$$

Σ is a real vector space and a module over the ring $\phi = \{f \mid f : M \to \mathbf{R},\ f \text{ is } C^0\}$.
For Σ it is used the WO^0-topology (the C^0-Whitney open topology).
The topological space $(\Sigma, WO^0 - top)$ is a topological additive abelian group.
The topological space $(\Sigma, WO^0 - top)$ is a real topological space if and only if M is a compact manifold.
A positive global section of the pair $[(E, p, M); K]$ is an element $\sigma \in \Sigma$, such that $\sigma(x) \in K(x)$, $(\forall x \in M)$. K_Σ is the set of all positive global sections of $[(E, p, M); K]$, i.e.

$$K_\Sigma \stackrel{def}{=} \{\sigma \mid \sigma \in \Sigma,\ \sigma(x) \in K(x),\ \forall X \in M\}.$$

K_Σ is a convex pointed closed cone with interior points of the real vector space Σ endowed with the WO^0-topology.

$$\operatorname{Int} K_\Sigma \overset{def}{=} \{\sigma \mid \sigma \in \Sigma,\ \sigma(x) \in \operatorname{Int} K(x),\ \forall x \in M\}.$$

The fundamental properties of the structure $(\Sigma; K_\Sigma)$ were studied in [10], Dan I. Papuc, 1992. They are essential different if M is compact or not. For example: $(\Sigma; K_\Sigma)$ *is a Krein space if and only if* M *is a compact space.*

b). Some properties of the structure (Σ, K_Σ):

1. K_Σ *is a generating cone for* Σ, i.e.

$$\forall \sigma \in \Sigma,\ \exists \zeta_1, \zeta_2 \in \operatorname{Int} K_\Sigma \mid \sigma = \zeta_1 - \zeta_2.$$

2. An **ordering** relation for Σ is defined:

$\sigma_1 \le \sigma_2 \overset{def}{\Leftrightarrow} \sigma_2 - \sigma_1 \in K_\Sigma;\ (\sigma_1, \sigma_2 \in \Sigma).$

$\sigma_1 \le \sigma_2 \Leftrightarrow \sigma_1(x) \le \sigma_2(x),\ (\forall x \in M).$

$(\Sigma; \le)$ *is an ordered vector space.*

3. For an element $\zeta \in \operatorname{Int} K_\Sigma$, the set of all ζ-measurable elements of Σ will be denoted by Σ_ζ, i.e.

$$\Sigma_\zeta \overset{def}{=} \{\sigma \mid \sigma \in \Sigma,\ \exists \lambda \in \mathbf{R},\ \lambda \ge 0,\ -\lambda\zeta \le \sigma \le \lambda\zeta\}.$$

We shall have:

3_0. Σ_ζ is a linear subspace of Σ.

3_1. $\Sigma_\zeta = \Sigma$ if and only if M is compact.

3_2. Σ_ζ has *a norm* $|\cdot|_\zeta : \sigma \in \Sigma \to |\sigma|_\zeta \in \mathbf{R}$, where

$$|\sigma|_\zeta \overset{def}{=} \min\{\lambda \mid \lambda \in \mathbf{R},\ \lambda \ge 0,\ -\lambda\zeta \le \sigma \le \lambda\zeta\}.$$

3_3. If Σ_c is the set of all sections with compact support, then

$$\Sigma_c = \bigcap_{\zeta \in \operatorname{Int} K_\Sigma} \Sigma_\zeta.$$

3_4. The set $\{\Sigma_\zeta \mid \forall \zeta \in \operatorname{Int} K_\Sigma\}$ is *a covering* of Σ.

4. For $\zeta \in \operatorname{Int} K_\Sigma$ an *equivalence relation on* Σ is defined by

$$\sigma_1 \sim \sigma_2 \overset{def}{\Leftrightarrow} \sigma_1 - \sigma_2 \in \Sigma_\zeta, \quad (\sigma_1, \sigma_2 \in \Sigma).$$

On every equivalence class $\sigma + \Sigma_\zeta$, **a distance** d_ζ is determined by the norm $|\cdot|_\zeta$ of Σ_ζ:

$$\forall \sigma_1, \sigma_2 \in \sigma + \Sigma_\zeta \mid d_\zeta(\sigma_1, \sigma_2) \overset{def}{=} |\sigma_1 - \sigma_2|_\zeta.$$

5. **Theorem**: *Let* ζ *be an arbitrary fixed element of* $\operatorname{Int} K_\Sigma$.

a). *For every equivalence class* $\sigma + \Sigma_\zeta$, *the metric topology determined by* d_ζ *and the topology of open ordered in* $\le$ *determined by* ζ *intervals coincide.*

b). *The* WO^0*-topology of* Σ *is the topology for which a base of neighbourhoods is the set of all open balls* $\{B(\sigma, \zeta, \varepsilon)\}$ *defined by means of the distances* d_ζ.

The equivalences classes $\sigma + \Sigma_\zeta$ are closed and open subsets of the topological space $(\Sigma, WO^0\text{-top})$.
If we consider only the algebraic structure of linear space on Σ and the cone K_Σ, then we can define open ordered intervals $(\sigma_0 - \varepsilon\zeta, \sigma_0 + \varepsilon\zeta)$ for $\forall\sigma_0 \in \Sigma$, $\forall\varepsilon \in \mathbf{R}$, $\varepsilon > 0$, $\forall\zeta \in \Sigma$, $(\forall x \in M, \zeta(x) \in \operatorname{Int} K(x))$. The topology defined by these open ordered intervals is exactly the WO^0-topology of Σ.

4 Positive operators on vector bundles endowed with cone fields

a). Operators; the cone of positive operators; the structure $(\Omega/\Omega_0; \mathbf{K})$
An operator $A : (E, p, M) \to (E', p', M)$ *is a continuous map* $A : E \to E'$ *for which*

$$A(p^{-1}(x)) \subset p'^{-1}(x), \quad (\forall x \in M).$$

Ω *is the set of all operators*, i.e

$$\Omega \stackrel{def}{=} \{A \mid A : E \to E', A \text{ is } C^0,\ (\forall x \in M)\ A(p^{-1}(x)) \subset p'^{-1}(x)\}.$$

Ω is *a real vector space* and *a module* over the ring of real continuous functions defined on M or on E.
For two structures $[(E, p, M); K_1]$ and $[E', p', M); K_2]$ **a positive operator**

$$A : [E, p, M); K_1] \to [(E'.p'.M); K_2]$$

is an operator $A : (E, p, M) \to (E'.p'.M)$ (i.e. $A \in \Omega$) for which

$$A(K_1(x)) \subset K_2(x), \quad \forall x \in M.$$

The subset of Ω *of all positive operators will be denoted by* K_Ω, i.e.

$$K_\Omega \stackrel{def}{=} \{A \mid A \in \Omega,\ A(K_1(x)) \subset K_2(x),\ (\forall x \in M)\}.$$

K_Ω is a convex generating cone of Ω.
For the linear subspace Ω_0 of Ω,

$$\Omega_0 \stackrel{def}{=} \{A \mid A \in \Omega,\ A(K_1(x)) = 0 \in K_2(x),\ \forall x \in M\}\ ,$$

we consider *the quotient linear space*

$$\Omega/\Omega_0 = \{[A] = A + \Omega_0 \mid \forall A \in \Omega\}.$$

We shall have $[A_1] = [A_2] \Leftrightarrow (\forall X \in K_1(p(X))),\ A_1(X) = A_2(X)$.
We consider the subset $\mathbf{K}$ of Ω/Ω_0,

$$\mathbf{K} \stackrel{def}{=} \{[A] \mid A \in K_\Omega\},$$

or equivalently

$$\mathbf{K} \stackrel{def}{=} \{[A] = A + \Omega_0 \mid A \in \Omega,\ (\forall x \in M)\ A(K_1(x)) \subset K_2(x)\}.$$

$\mathbf{K}$ *is a convex pointed generating cone of the linear space* Ω/Ω_0. *The interior of the cone* $\mathbf{K}$ *is*

$$\operatorname{Int}\mathbf{K} \stackrel{def}{=} \{[B] \mid [B] \in \Omega/\Omega_0,\ B(\operatorname{Int} K_1(x)) \subset \operatorname{Int} K_2(x),\ \forall x \in M\}.$$

Obviously, $\operatorname{Int}\mathbf{K} \subset \mathbf{K}$.
For two structures $[(E, p, M); K_1]$ and $[(E', p', M); K_2]$ we consider the structure $(\Omega/\Omega_0; \mathbf{K})$.
b). An ordering relation for the structure $(\Omega/\Omega_0; \mathbf{K})$.
For the linear space Ω/Ω_0, an ordering relation is defined by:

$$[A_1], [A_2] \in \Omega/\Omega_0 \mid [A_1] \le [A_2] \stackrel{def}{\Leftrightarrow} [A_2] - [A_1] \in \mathbf{K}.$$

Obviously

$$\begin{gathered}[A_1] \le [A_2] \stackrel{def}{\Leftrightarrow} [A_2] - [A_1] \in \mathbf{K} \Leftrightarrow [A_2 - A_1] \in \mathbf{K} \Leftrightarrow \\ \Leftrightarrow (\forall X \in K_1(x)) \quad (A_2 - A_1)(X) \in K_2(x) \Leftrightarrow \\ \Leftrightarrow (\forall X \in K_1(x)) \quad A_1(X) \le A_2(X) \Leftrightarrow \\ \Leftrightarrow (\forall X \in K_1(x)) \quad A_1(X) - A_2(X) \in K_2(x).\end{gathered}$$

The pair $(\Omega/\Omega_0; \le)$ *is an ordered vector space, directed on both sides.*
An ordered interval centered in $[A^*] \in \Omega/\Omega_0$, determined by an element $[B] \in \operatorname{Int}\mathbf{K}$, *is the set*

$$\begin{gathered}\ll [A^*] - \varepsilon[b],\ [A^*] + \varepsilon[B] \gg \stackrel{def}{=} \{|A| \in \Omega/\Omega_0 \mid \exists \varepsilon, \varepsilon_1 \in \mathbf{R},\ 0 < \varepsilon_1 << \varepsilon : \\ [A^*] - \varepsilon_1[B] \le [A] \le [A^*] + \varepsilon_1[B]\}.\end{gathered}$$

The set of all ordered open intervals determined by the same element $[B] \in \operatorname{Int}\mathbf{K}$ *is a base of a topology of* Ω/Ω_0, *denoted by* $\tau_{[B]}$.
c) Norms and distances for the structure $(\Omega/\Omega_0; \mathbf{K})$.
An element $[A] \in \Omega/\Omega_0$ *is called* $[B]$-**measurable**, $[B] \in \operatorname{Int}\mathbf{K}$, *if there is* $\lambda \in \mathbf{R}$, $\lambda > 0$ *such that* $-\lambda[B] \le [A] \le \lambda[B]$.
The set of all elements of Ω/Ω_0 which are $[B]$-**measurable** will be denoted by $\Delta_{[B]}$. This set is a real linear subspace of Ω/Ω_0.
The set Δ *of all linear spaces* $\Delta_{[B]}$,

$$\Delta = \{\Delta_{[B]} \mid [B] \in \operatorname{Int}\mathbf{K}\},$$

is a covering of Ω/Ω_0.
On the set Δ we consider an **ordering relation**

$$\Delta_{[B']}, \Delta_{[B'']} \in \Delta \mid \Delta_{[B']} \le \Delta_{[B'']} \stackrel{def}{\Leftrightarrow} \Delta_{[B']} \subset \Delta_{[B'']}.$$

This ordering relation is directed on both sides.
The following relations

$$\Delta_{[B']} \subset \Delta_{[B'']} \Leftrightarrow [B'] \in \Delta_{[B'']} \Leftrightarrow \exists \lambda \in \mathbf{R},\ \lambda > 0,\ [B'] \leq \lambda[B'']$$

hold.
The map $|\cdot|_{[B]} : \Delta_{[B]} \to \mathbf{R}$, ($[B] \in \operatorname{Int}\mathbf{K}$), where

$$(\forall [A] \in \Delta_{[B]}),\ \ |[A]|_{[B]} \stackrel{def}{=} \min\{\lambda \mid \lambda \in \mathbf{R},\ \lambda \geq 0, -\lambda[B] \leq [A] \leq \lambda[B]\}$$

is a monotone norm on the linear space $\Delta_{[B]}$.
For the linear subspace $\Delta_{[B]} \subset \Omega/\Omega_0$ we consider the linear quatient $(\Omega/\Omega_0)/\Delta_{[B]}$. An element of this linear quatient determined by an arbitrary $[A]$ ($[A] \in \Omega/\Omega_0$) will be the class of equivalence $[A] + \Delta_{[B]}$.
For the class $[A] + \Delta_{[B]}$, the map $d_{[B]} : ([A] + \Delta_{[B]}) \times ([A] + \Delta_{[B]}) \to \mathbf{R}$, where

$$d_{[B]}([A'] + \Delta_{[B]}, [A''] + \Delta_{[B]} \stackrel{def}{=} |[A'] - [A'']|_{[B]} \in \mathbf{R}\ ,$$

is *a distance* on $[A] + \Delta_{[B]}$.

5 The lifting of the cone field K for tensor bundles associated to $[(E, p, M); K]$

An important property of the structure $[(E, p, M); K]$ is **the existence of a "natural" lifting of** K for all tensor bundles associated to (E, p, M).
a). For a vector bundle (E, p, M) it is considered the dual vector bundle (E^*, p^*, M) where

$$E^* = \{\xi_x \mid \xi_x : E_x \to \mathbf{R},\ \forall x \in M,\ \xi_x \text{ is a linear map}\}.$$

To $[(E, p, M); K]$ it is associated $[(E^*, p^*, M); K^*]$, where

$$K^* : x \in M \to K^*(x) \subset E^*_x \subset E^*,$$

$$K^*(x) = \{\xi_x \mid \xi_x \in E^*_x,\ \xi_x(X_x) \geq 0,\ \forall X_x \in K(x)\}.$$

The pair $[(E^*, p^*, M); K^*]$ *satisfies the axioms* A_1 *and* A_2.
A linear functional $\zeta_x \in E^*_x$ belongs to $\operatorname{Int} K^*(x)$ if and only if $\zeta_x(X_x) > 0$ for every $X_x \in K(x)$, $X_x \neq 0$.
b). A $\binom{q}{p}$-tensor t_x is a polylinear (with respect to $\mathbf{R}$) map

$$t_x : E_x \times E_x \times \cdots \times E_x \times E^*_x \times E^*_x \times \cdots \times E^*_x \to \mathbf{R}.$$

By means of the cone fields K and K^*, to the vector bundle of $\binom{q}{p}$-tensor, (E^q_p, p, M), a cone field

$$K^q_p : x \in M \to K^q_p(x) \subset {E_p}^q{}_x \subset {E_p}^q,$$

where

$$K_p^q(x) \stackrel{def}{=} \{t_x \mid t_x \in E_{px}^q,\ t_x(X_1, X_2, \dots, X_p, \xi^1, \xi^2, \dots, \xi^q) \geq 0,$$
$$\forall X_i \in K(x),\ \forall \xi^j \in K^*(x);\ i = 1, \dots, p;\ j = 1, \dots, q\},$$

is associated.

The pair $[(E_p^q, p, M); K_p^q]$ *satisfies the axioms* A_1 *and* A_2.

Conclusion: To an arbitrary structure $[(E, p, M); K]$ it is uniquely associated, in a "natural manner", the structures $[(E_p^q, p, M); K_p^q]$ and $(\Sigma_p^q, K_{p\Sigma}^q)$, for all $p, q \in \mathbf{N}$, $p + q > 0$.

The $\mathrm{Int}\, K_p^q(x)$ in the topological space E_{px}^q is the set

$$\mathrm{Int}\, K_p^q(x) = \{t_x \mid t_x \in E_{px}^q,\ t_x(X_1, X_2, \dots, X_p, \xi^1, \xi^2, \dots, \xi^q) > 0,$$
$$\forall X_i \in K(x),\ \forall \xi^j \in K^*(x);\ i = 1, \dots, p;\ j = 1, \dots, q\}.$$

If the vectors $Z \in \mathrm{Int}\, K(x)$ and $\zeta \in \mathrm{Int}\, K^*(x)$ are fixed, then a tensor $t_x \in \mathrm{Int}\, K_p^q(x)$ is uniquely defined by

$$t_x \stackrel{def}{=} Z \otimes Z \otimes \dots \otimes Z \otimes \zeta \otimes \zeta \otimes \dots \otimes \zeta.$$

References

[1] Liana David, *About the positive local operators on the sections of a regular vector bundle*, Proceedings of the 25th National Conference of Geometry and Topology, Iassy, Romania, September 18–22, 1995.

[2] Liana David, *Spectral properties of the positive operators defined on a vector bundle endowed with a field of cones*, Sem. de Geom. şi Top. Univ. din Timişoara, nr. 109, 1995, An. Univ. Timişoara.

[3] Liana David and Dan I. Papuc, *About the positive operators on vector bundles*, Riv. Mat. di Parma (5) **5** (1996), 55–64.

[4] Liana David and Dan I. Papuc, *About the composition of positive operators on vector bundles* (to appear, 1999).

[5] Mark A. Krasnosel'skij, *Positive solutions of operators equations*, in Russian in 1962 and in English at Groningen Noordhoff in 1964.

[6] Mark A. Krasnosel'skij, Je. A. Lifshits, A. V. Sobolev, *Positive linear systems. The method of positive operators*, Heldermann Verlag Berlin, 1989.

[7] Dan I. Papuc, *Partial orderings on differentiable manifolds*, An. Univ. Timişoara, Seria Matematică, vol. XXVI, f.1, 55–73, 1988.

[8] Dan I. Papuc, *On the geometry of a differentiable manifold with a partial ordering (or with a field of tangent cones)*, An. Univ. Timişoara, seria Matematică, vol. XXVI, f.2, 39–48, 1988.

[9] Dan I. Papuc, *Ordered tangent structures associated to a differentiable manifold with a regular field of tangent cones*, Bull. Math. de la Soc. Sci. Math. de Roumanie, t. 34, (82), nr. 2, 173–179, 1990.

[10] Dan I. Papuc, *Field of cones and positive operators on a vector bundle*, An. Univ. Timişoara, Seria Matematică, vol. XXX, fasc. 1, 39–58, 1992.

[11] Dan I. Papuc, *Field of cones on a tensor bundle*, Proceedings of the 24 National Conference of Geometry and Topology, Timişoara, Romania, July 5–9 1994, vol. II, pp. 237–242.

[12] Dan I. Papuc, *Fields cones and positive operators on regular vector bundles*, (Research-expository paper, 25^{th} Conf. Nat. Geom. Iaşi, 1995), An. Şt. Univ. "Al. I. Cuza" Iaşi, t. XLII, Supliment, s I-a, Matematică, 1996, p. 39–58.

[13] Dan I. Papuc, *A new geometry of a Lorentzian manifold*, Publicationes Mathematicae, Debrecen, tomus 52 (1998), fasc. 1–2, p. 145–158.

[14] Dan I. Papuc, *On the geometry of the bundle of symmetric* 2*-times covariant tensors tangent to a differentiable manifold*, An. Şt. Univ. "Al. I. Cuza" Iaşi, t. XLIII, s I-a, Matematica, 1997, f.2, p. 249–258.

[15] Dennis Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Inventiones Math., 36, (1976) 225–255.

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