An Optimal Control Problem on the Heisenberg Lie Group $H(3)$

Camelia Pop

Abstract

An optimal control problem on the Heisenberg Lie group $H(3)$ is discussed and some of its geometrical and dynamical properties are pointed out.

1 Introduction

In the last time there was a great deal of interest in the study of control systems on matrix Lie groups in connection with their deep applications in robotics, classical mechanics and engineering.

The goal of our paper is to study such control system on the Heisenberg Lie group $H(3)$.

2 The Heisenberg Lie group $H(3)$ and its Lie algebra

The Heisenberg group $H(3)$ is given by the set of all matrices of type:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

where $a, b, c \in \mathbf{R}$,

It is obvious that $H(3)$ is a closed subgroup of $\mathcal{M}_{3\times 3}(\mathbf{R})$, so it is a Lie group.

Theorem 1 *Its Lie algebra has the following form:*

$$h(3) = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbf{R} \right\}.$$

Proof. Indeed

$$c : t \in \mathbf{R} \to c(t) = \begin{bmatrix} 1 & \alpha t & \beta t \\ 0 & 1 & \gamma t \\ 0 & 0 & 1 \end{bmatrix} \in H(3)$$

is a curve in $H(3)$ with

$$c(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$\frac{d}{dt}c(t)|_{t=0} \in h(3) \ ,$$

i. e.

$$\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \in h(3) \ .$$

Since dim $h(3) = 3$ we can deduce immediately that

$$h(3) = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbf{R} \right\}$$

as required.□

Theorem 2 *The constants of structure of the Lie algebra $h(3)$ are given by:*

$$c_{13}^2 = 1$$

$$c_{31}^2 = -1$$

and the others:

$$c_{ij}^k = 0 \ .$$

Proof. Let us recall that the bracket operation on $h(3)$ is given by the Table 1:

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	0	A_2
A_2	0	0	0
A_3	$-A_2$	0	0

where $\{A_1, A_2, A_3\}$ is the canonical basis of $h(3)$, i.e.:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \ .$$

Then we have successively:

$$[A_1, A_1] = c_{11}^k A_k$$

so

$$c_{11}^1 = c_{11}^2 = c_{11}^3 = 0;$$

$$[A_1, A_2] = c_{12}^k A_k$$

so

$$c_{12}^1 = c_{12}^2 = c_{12}^3 = 0;$$

$$[A_1, A_3] = c_{13}^k A_k$$

so

$$c_{13}^1 = 0, c_{13}^2 = 1, c_{13}^3 = 0;$$

$$[A_2, A_3] = c_{23}^k A_k$$

so

$$c_{23}^1 = c_{23}^2 = c_{23}^3 = 0;$$

$$[A_3, A_3] = 0$$

so

$$c_{33}^1 = c_{33}^2 = c_{33}^3 = 0. \qquad \blacksquare$$

As a consequence we obtain:

Theorem 3 *The minus Lie-Poisson structure on $h(3)^*$ is given by the matrix:*

$$\Pi_- = \begin{bmatrix} 0 & 0 & -P_2 \\ 0 & 0 & 0 \\ P_2 & 0 & 0 \end{bmatrix} .$$

Remark 1 It is easy to see that the function C given by

$$C(P_1, P_2, P_3) = P_2$$

is a Casimir of our configuration $(h(3), \Pi_-)$, i.e.

$$(\nabla C)^t \cdot \Pi_- = 0 .$$

Theorem 4 *There exist the following two types of controlable drift-free left invariant systems on $H(3)$, namely:*

$$\dot{X} = X \cdot (A_1 u_1 + A_2 u_2 + A_3 u_3) \tag{2.1}$$

$$\dot{X} = X \cdot (A_1 u_1 + A_3 u_3) . \tag{2.2}$$

Proof. The proof is a consequence of the Table 1 and Chow's theorem. $\blacksquare$

3 An optimal control problem on $H(3)$

Let

$$J(u_1,u_2,u_3)=\frac{1}{2}\int_0^{t_f}(c_1u_1^2+c_2u_2^2+c_3u_3^2)\ dt$$

respectively

$$J(u_1,u_2)=\frac{1}{2}\int_0^{t_f}(c_1u_1^2+c_3u_3^2)\ dt$$

$(c_1,c_2,c_3>0$, resp. $c_1,c_3>0)$ be the cost function.

Then the problem which we intend to solve is the following: find u_1,u_2,u_3 (resp. u_1,u_2) that minimize J and steer the system (2.1) (resp. (2.2)) from $X=0$ at $t=0$ to $X=X_f$ at $t=t_f$.

We have the following results:

Theorem 5 *The optimal controls of the above problem for our system (2.1) are given by*

$$\begin{cases} u_1=\frac{1}{C_1}P_1\ , \\ u_2=\frac{1}{C_2}P_2\ , \\ u_3=\frac{1}{C_3}P_3\ , \end{cases}$$

where P_i are solutions of the system:

$$\begin{cases} \dot{P}_1=-\frac{1}{C_3}P_2P_3\ , \\ \dot{P}_2=0\ , \\ \dot{P}_3=\frac{1}{C_1}P_1P_2\ . \end{cases} \tag{3.1}$$

Proof. Let us take the extended Hamiltonian H given by:

$$H=P_1u_1+P_2u_2+P_3u_3-\frac{1}{2}(c_1u_1^2+c_2u_2^2+c_3u_3^2)\ .$$

Then using the maximum principle, we have the conditions:

$$\begin{cases} \frac{\partial H}{\partial u_1}=0\ , \\ \\ \frac{\partial H}{\partial u_2}=0\ , \\ \\ \frac{\partial H}{\partial u_3}=0\ . \end{cases}$$

which lead us to:

$$\begin{cases} P_1=c_1u_1\ , \\ P_2=c_2u_2\ , \\ P_3=c_3u_3\ , \end{cases}$$

and so the reduced Hamiltonian (or the optimal Hamiltonian) is given by:

$$H=\frac{1}{2}\left[\frac{P_1^2}{C_1}+\frac{P_2^2}{C_2}+\frac{P_3^2}{C_3}\right]\ . \tag{3.2}$$

It follows that the reduced Hamilton equations have the following expressions:

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -P_2 \\ 0 & 0 & 0 \\ P_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{P_1}{C_1} \\ \frac{P_2}{C_2} \\ \frac{P_3}{C_3} \end{bmatrix} = \begin{bmatrix} -\frac{P_2 P_3}{C_3} \\ 0 \\ \frac{P_1 P_2}{C_1} \end{bmatrix}$$

as required.□

Since $P_2 = K_2$ it is easy to see that the reduced Hamilton's equations can be put in the equivalent form:

$$\begin{cases} \dot{P}_1 = -\frac{K_2}{C_3} P_3 \ , \\ \\ \dot{P}_3 = \frac{K_2}{C_1} P_1 \ , \end{cases}$$

and then we have:

Theorem 6 *The controls u_1, u_3 are given by sinusoids. More exactly:*

$$u_1 = \frac{l\sqrt{C_1}}{C_1} \cos\left(\frac{k_2}{\sqrt{C_1 C_3}} t + k \right) \ ,$$

$$u_3 = \frac{l\sqrt{C_3}}{C_3} \sin\left(\frac{k_2}{\sqrt{C_1 C_3}} t + k \right) \ .$$

Proof. The reduced Hamiltonian (3.2) is obviously a constant of motion, so:

$$\frac{P_1^2}{C_1} + \frac{P_3^2}{C_3} = l^2 \ .$$

If we take now:

$$\begin{cases} P_1 = l\sqrt{C_1} \cos\theta \ , \\ P_3 = l\sqrt{C_3} \sin\theta \ , \end{cases}$$

then:

$$\dot{P}_1 = -\frac{\sqrt{C_1}}{\sqrt{C_3}} P_3 \dot{\theta}$$

and so:

$$\dot{\theta} = -\frac{\sqrt{C_3}}{\sqrt{C_1}} \cdot \frac{\dot{P}_1}{P_3} = -\frac{\sqrt{C_3}}{\sqrt{C_1}} \left(\frac{k_2}{C_3} \right) = \frac{k_2}{\sqrt{C_1 C_3}} \ .$$

It follows that:

$$\theta = \frac{k_2}{\sqrt{C_1 C_3}} t + k$$

and then our assertion follows immediately.□

In the same manner we have:

Theorem 7 *The optimal control for the second system (2.2) are given by:*

$$\begin{cases} u_1 = \frac{1}{C_1} P_1 \ , \\ u_2 = 0 \ , \\ u_3 = \frac{1}{C_3} P_3 \ , \end{cases}$$

where P_i are solutions of the system:

$$\left\{\begin{array}{l} \dot{P}_1 = -\frac{1}{C_3} P_2 P_3 \ , \\ \dot{P}_2 = 0 \ , \\ \dot{P}_3 = \frac{1}{C_1} P_1 P_2 \ . \end{array}\right. \tag{3.3}$$

Proof. Let us take the extended Hamiltonian H given by

$$H = P_1 u_1 + P_3 u_3 - \frac{1}{2}(c_1 u_1^2 + c_3 u_3^2).$$

Then, using the Maximum principle, we have the conditions

$$\left\{\begin{array}{l} \frac{\partial H}{\partial u_1} = 0 \ , \\ \\ \frac{\partial H}{\partial u_2} = 0 \ , \\ \\ \frac{\partial H}{\partial u_3} = 0 \ , \end{array}\right.$$

which lead us to

$$\left\{\begin{array}{l} P_1 = c_1 u_1 \\ P_3 = c_3 u_3 \end{array}\right. .$$

So, the reduced Hamiltonian is given by:

$$H = \frac{1}{2}\left[\frac{P_1^2}{C_1} + \frac{P_3^2}{C_3}\right] . \tag{3.4}$$

It follows that the reduced Hamilton equations have the following expressions:

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -P_2 \\ 0 & 0 & 0 \\ P_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{P_1}{C_1} \\ \frac{P_2}{C_2} \\ \frac{P_3}{C_3} \end{bmatrix} = \begin{bmatrix} -\frac{P_2 P_3}{C_3} \\ 0 \\ \frac{P_1 P_2}{C_1} \end{bmatrix}$$

as required.□

Since $P_2 = K_2$ it is easy to see that the reduced Hamilton equations can be put in the equivalent form:

$$\left\{\begin{array}{l} \dot{P}_1 = -\frac{K_2}{C_3} P_3 \\ \\ P_2 = K_2 \\ \\ \dot{P}_3 = \frac{K_2}{C_1} P_1 \end{array}\right. .$$

Theorem 8 *The controls u_1, u_3 are given by:*

$$u_1 = \frac{l\sqrt{C_1}}{C_1} \cos\left(\frac{k_2}{\sqrt{C_1 C_3}} t + k\right) \ ,$$

$$u_3 = \frac{l\sqrt{C_3}}{C_3} \sin\left(\frac{k_2}{\sqrt{C_1 C_3}} t + k\right) \ .$$

Proof. The reduced Hamiltonian (3.4) is obviously a constant of motion, so:

$$\frac{P_1^2}{C_1} + \frac{P_3^2}{C_3} = l^2 .$$

If we take now

$$\begin{cases} P_1 = l\sqrt{C_1}\cos\theta \\ P_3 = l\sqrt{C_3}\sin\theta \end{cases} ,$$

then

$$\dot{P}_1 = -\frac{\sqrt{C_1}}{\sqrt{C_3}} P_3 \dot{\theta}$$

and so

$$\dot{\theta} = -\frac{\sqrt{C_3}}{\sqrt{C_1}} \cdot \frac{\dot{P}_1}{P_3} = -\frac{\sqrt{C_3}}{\sqrt{C_1}} \left(\frac{k_2}{C_3} \right) = \frac{k_2}{\sqrt{C_1 C_3}} .$$

It follows that:

$$\theta = \frac{k_2}{\sqrt{C_1 C_3}} t + k$$

and then our assertion follows immediately.□

References

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"Politehnica" University of Timişoara
Piaţa Horaţiu, Nr. 1
1900 Timişoara
România