On Algebra Bundles and Pseudo-morphisms

Marcela Popescu and Paul Popescu

Abstract

The aim of this paper is to define the associative algebra bundles which have the fibre a real associative algebra. The real, complex and quaternionic models for the fiber of the algebra bundles are considered as examples. Pseudo-morphisms of algebras and of associative algebra bundles are defined as generalisations of morphisms. Some further results concerning these pseudo-morphisms are also stated.

1 Introduction

For a differentiable manifold M, the commutative real algebra $\mathcal{F}(M)$ of real functions can be considered in a functorial manner. This algebra play an important role in the construction of the differential geometry objects on M. Replacing this algebra with a non-commutative algebra, for instance the algebra $\mathcal{F}(M) \otimes \mathcal{M}_k(I\!R)$ in [1], some analogous objects of the classical $\mathcal{F}(M)$ case can be considered: derivations of this algebra, Riemannian metrics on derivations, etc.

The aim of this paper is to study some aspects concerning the replacement of the commutative real algebra $\mathcal{F}(M)$, which is composed in fact by global sections of the trivial algebra bundle with total space $M \times I\!R$, with another algebra bundle, which has the fibre an associative algebra only . Specifically, we consider in the paper the cases when the algebras are the fields which are finite dimensional algebras. It means, according to a theorem of Frobenius, that the fibres are $I\!R$, $\mathbf{C}$ and $I\!K$ (real, complex and quaternion fields). Besides the algebra morphisms, the pseudo-morphisms of algebras are also considered in the last section. Further properties of algebra bundles are considered in [6].

2 Algebra bundles

The algebras considered throughout the paper as algebra bundles models are unitary, associative and finite dimensional real vector spaces. They are denoted as $\mathcal{A}$, $\mathcal{A}'$ etc. The definitions can be easily adapted for infinite dimensional algebras, for complex algebras or for graded algebras, but the concrete cases considered here are only $I\!R$, $\mathbf{C}$ and $I\!K$. The other cases will be presented in [6] and elsewhere.

For a real algebra $\mathcal{A}$ as above, we denote by $\mathbf{Z}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \cdot b = b \cdot a, (\forall) b \in \mathcal{A}\}$, the center of the algebra, that is a commutative subalgebra of $\mathcal{A}$. We denote as $\mathcal{D}er(\mathcal{A})$ the $\mathbf{Z}(\mathcal{A})$-module of the derivations in algebra $\mathcal{A}$, i.e. the set of $\mathbf{k}$-linear maps $D : \mathcal{A} \to \mathcal{A}$ which have the property $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$, $(\forall) a, b \in \mathcal{A}$. If $X \in \mathcal{D}er(\mathcal{A})$ and $a \in \mathcal{A}$, we use also the notation $X(a) = [X, a]$.

The algebra morphisms $f : \mathcal{A}' \to \mathcal{A}$ considered below are of two types:

1. The *first type morphisms* are the morphisms that preserve the center, i.e. $f(\mathbf{Z}(\mathcal{A}')) \subset \mathbf{Z}(\mathcal{A})$.
2. The *second type morphisms* are the morphisms provided that there are commutative algebra morphisms $f_0 : \mathbf{Z}(\mathcal{A}) \to \mathbf{Z}(\mathcal{A}')$ which enjoy the property $i = f \circ i' \circ f_0$, where $i : \mathbf{Z}(\mathcal{A}) \to \mathcal{A}$ and $i' : \mathbf{Z}(\mathcal{A}') \to \mathcal{A}'$ are the inclusion morphisms.

For a second type morphism f as above, the commutative algebra morphism $f_0 : \mathbf{Z}(\mathcal{A}) \to \mathbf{Z}(\mathcal{A}')$ is injective, thus $f_0\,(\mathbf{Z}(\mathcal{A})) \subset \mathbf{Z}(\mathcal{A}')$ is a commutative subalgebra.

We say that a subalgebra $\mathcal{A}'$ of an algebra $\mathcal{A}$ is a *first* or a *second type* subalgebra if the inclusion $I : \mathcal{A}' \to \mathcal{A}$ has the first or the second type.

For example, the subalgebra $I\!R \subset \mathbf{C}$ is of the first type, but not of the second. The subalgebra $I\!R \subset I\!K$ is of the first and second type. The subalgebra $\mathbf{C} \subset I\!K$ is of the second type, with f_0 the inclusion algebra morphism

$$\mathbf{Z}(I\!K) = I\!R \to \mathbf{C} = \mathbf{Z}(\mathbf{C}),$$

but it is not of the first type.

Note that an isomorphism of algebras is of first and second type.

The differentiable manifolds considered in what follows belong to the class C^1 at least.

Definition 1 Let $\xi = (E, \pi, M)$ be a real vector bundle which has as a model a real, unitary and associative algebra $\mathcal{A}$. We say that ξ is an *associative algebra bundle (a.a.b.)* if there is an open cover $\{U_i\}_{i\in I}$ of the base M and a system of structural functions (or cocycles) $\{g_{ij}\}$ of ξ , each g_{ij} defined on $U_i \cap U_j \neq \emptyset$, and $g_{ij} \in Aut(\mathcal{A})$.

Let $\xi = (E, \pi, M)$ be an a.a.b. We denote by$End(\xi)$ and $Aut(\xi)$ the algebra bundle of endomorphisms and the group automorphisms respectively of the a.a.b. ξ.

If $\xi' \xrightarrow{i} \xi$ is the inclusion of a vector subbundle and the induced structure on each fibre is a subalgebra structure, then we say that ξ' is a *subbundle* of the *associative algebra bundle (s.a.a.b.)*.

Consider now an a.a.b. $\xi = (E, \pi, M)$ which has as a model a real, unitary and associative algebra $\mathcal{A}$. The following property follows easily, using the fact that the cocycles $g_{ij} \in Aut(\mathcal{A})$.

Proposition 1 *For every local vector chart (U, φ), where $\varphi : U \times \mathcal{A} \to \pi^{-1}(U)$, let us denote, for every $x \in U$, as $t_x : \mathcal{A} \to \pi^{-1}(x)$, defined by $t_x(a) = \varphi(x, a)$.*

Then the fiber $E_x = \pi^{-1}(x)$ has a real, unitary and associative algebra structure, so that t_x is an algebra isomorphism.

Corrolary 1 *The set of sections $\Gamma(\xi)$ has a real, unitary and associative algebra structure.*

Corrolary 2 *The a.a.b. ξ contain as an s.a.a.b. $\mathbf{Z}(\xi) = (\mathbf{Z}(E), \pi_Z, M)$, where in every $x \in M$ we have $\mathbf{Z}_x(E) = \mathbf{Z}(E_x)$. The algebra model of $\mathbf{Z}(\xi)$ is $Z(\mathcal{A})$.*

Definition 2 We call the a.a.b. $\mathbf{Z}(\xi)$ as the *center* of ξ.

Obviously we have $\Gamma(\mathbf{Z}(\xi)) = Z(\Gamma(\xi))$ and if $\mathcal{A}$ is a commutative algebra, then $\xi = \mathbf{Z}(\xi)$.

If $\xi = (E, \pi, M)$ is an a.a.b. and $x_0 \in M$, then the *germ* $[s]$ of $s \in \Gamma(\xi)$ in x_0 is an equivalence class according to the equivalent relation on $\Gamma(\xi)$, defined by $s_1 \overset{x_0}{\sim} s_2$ if there is an open neighborhood $U \ni x_0$, such that $s_{1|U} = s_{2|U}$. We can associate with every germ $[s]$ its value in x_0, $s(x_0) \in E_{x_0}$.

Definition 3 If $\xi = (E, \pi, M)$ is an a.a.b. and $x_0 \in M$, then the set of equivalence classes $\tilde{E}_{x_0} = \Gamma/\overset{x_0}{\sim}$ is called *the algebra of section germs* in x_0.

An x_0*-derivation* on $\tilde{E}_{x_0}$ is a map $D : \tilde{E}_{x_0} \to E_{x_0}$ which fulfills the relations:

1. D is $\mathbb{R}$-linear map.
2. $D([s] \cdot [t]) = D([s]) \cdot t(x_0) + s(x_0) \cdot D([t])$, $(\forall)[s], [t] \in \tilde{E}_{x_0}$.

We denote as $\mathcal{D}^1(x_0, \xi)$ the real vector space of the x_0-derivations and we call it as the *tangent space* in x_0 to the a.a.b. ξ.

It is proved in [6] that $\mathcal{D}^1(x_0, \xi)$ is a finite dimensional space and the union of these spaces over the base M is the total space of a vector bundle $\mathcal{D}^1(\xi)$ on the base M.

The cases when the fibre type $\mathcal{A}$ is a field are considered in the following two subsections. According to a well-known Theorem of Frobenius every real and finite dimensional algebra, which is associative and without divisors of zero, is one of the fields $\mathbb{R}$, $\mathbf{C}$ (if the algebra is commutative) or $\mathbb{K}$ (if the algebra is non-commutative).

2.1 The real and the complex cases

If $\mathcal{A} = \mathbb{R}$, then $g_{ij} = id_{\mathbb{R}}$, since the only automorphism of the field $\mathbb{R}$ is the identity. It follows that in this case, for every manifold M, the a.a.b. ξ is isomorphic with the trivial algebra bundle $(M \times \mathbb{R}, \pi_1, M)$, and $\Gamma(\xi) \cong \mathcal{F}(M)$ is the real algebra of functions on M.

We suppose from now on in this subsection that $\mathcal{A} = \mathbf{C}$ (the complex field). The set of automorphisms $Aut(\mathbf{C})$ has two elements: the identity and the complex conjugation. As we shall see below, not all the a.a.b. in this case are trivial bundles.

The tangent bundles τM^2 and τK^2, where M^2 is the Möbius band and K^2 is the Klein bottle, are a.a.b.. Indeed, the proper discontinuous actions:

$$a_1 : \mathbb{Z}^2 \times \mathbb{R}^2,\ a_1\left((m,n),(x,y)\right) = \left(x+m, (-1)^n y\right),$$

$$a_2 : \mathbb{Z}^2 \times \mathbb{R}^2,\ a_2\left((m,n),(x,y)\right) = \left(x+m, (-1)^n y + n\right)\ ,$$

have as orbit spaces the real manifolds τM^2 and τK^2 respectively. It can be easy seen that the local trivialisations of these manifolds have as Jacobians $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or

$\bar{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, thus the tangent vector bundles τM^2 and τK^2 are a.a.b., which has as fibre type $\mathbf{C}$.

For every a.a.b. $\xi = (E, \pi, M)$ which has $\mathbf{C}$ as a model, we can consider the points $w \in E_x$ which enjoy the property $w = Re\,(w)$; they define an s.a.a.b., denote as $Re(\xi)$, which has the total space $Re(E)$; we can define analogously the vector subbundle $Im(\xi)$, which is not an s.a.a.b.. The a.a.b. $Re(\xi)$ has the fibre type $I\!R$. Thus, according to the first example in this subsection, $Re(\xi)$ is isomorphic with the trivial bundle $(M \times I\!R, \pi_1, M)$. The map $E_x \ni w \to Re(w) = \dfrac{w + \bar{w}}{2} \in Re(E)_x$ is a left splitting of the inclusion $Re(\xi) \to \xi$, which is an m.a.a.b.s. ($I\!R$ and $\mathbf{C}$ are considered as real algebras). The complex conjugations on fibers induces an involutive automorphism on ξ. On the fibers E_x, $x \in M$, of ξ, the Riemannian metric $g_x(z_1, z_2) = \frac{1}{2}(z_1 \cdot \bar{z}_2 + \bar{z}_1 \cdot z_2)$ can be considered. The vector subbundles $Re(\xi)$ and $Im(\xi)$ are orthogonal and supplementary according to this metric. While the vector bundle $Re(\xi)$ is trivial (generated by the unity on each fiber), vector bundle $Im(\xi)$ is not always trivial. For example, in the case of the bundle τM^2, it is the vertical bundle $v\tau M^2$, which is isomorphic with $\pi^* \mathcal{M}^2$, where $\mathcal{M}^2 = (M^2, \pi, S^1)$ is the Möbius vector bundle over the base S^1.

2.2 The quaternion case

If $\mathcal{A} = I\!H$ is the field of quaternions, the set of automorphisms $Aut(I\!H)$ contains exactly the adjunction automorphisms, each defined for an $\alpha \in I\!H$ by $Ad_\alpha(w) = \alpha \cdot w \cdot \alpha^{-1}$. It is well-known that $\mathbf{Z}(I\!H) = I\!R$. As in the complex case, the s.a.a.b. $Re(\xi)$ can be considered. It is exactly $\mathbf{Z}(\xi)$, and is a trivial a.a.b. with the fiber $I\!R$. On the fibers E_x, $x \in M$, of ξ, the Riemannian metric $g_x(z_1, z_2) = \frac{1}{2}(z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1)$ can be considered. The orthogonal projection according to this metric allows to construct a left splitting $g : Re(\xi) \to \xi$ of the inclusion $i : Re(\xi) \to \xi$, which is thus an c.m.a.a.b. which splits (see the Definition 6 below for comorphisms which split). Note that $I\!R$ and $I\!H$ are considered as real algebras.

3 Algebra pseudo-morphisms and associative algebra bundle

Definition 4 If $\mathcal{A}'$ and $\mathcal{A}$ are real algebras, an algebra pseudo-morphism is a linear map $f : \mathcal{A}' \to \mathcal{A}$ which enjoys the property that there is $\alpha \in Z(\mathcal{A})$ such that:

$$\alpha \cdot f(x' \cdot y') = f(x') \cdot f(y'),\ (\forall) x', y' \in \mathcal{A}'.$$

Definition 5 If $\xi' = (E', \pi', M')$ and $\xi = (E, \pi, M)$ are associative algebra bundles which have as models the real associative unitary and real algebras $\mathcal{A}'$ and $\mathcal{A}$ respectively, and $f_0 : M \to M'$, Then an *f_0-pseudo-comorphism of a.a.b. (f_0-p.c.m.a.a.b.)* is an f_0-comorphism of vector bundles $f : \xi' \to \xi$ which induces, for every $x \in M$, an algebra pseudo-morphism $f_x : E'_{f_0(x)} = \pi'^{-1}(f_0(x)) \to \pi^{-1}(x) = E_x$. We call the section $\alpha \in \Gamma(Z(\xi))$, *central section* of the f_0-pseudo-comorphism f.

If $\alpha = 1$ we find the definition of the f_0-*comorphism of a.a.b.* (f_0-*c.m.a.a.b.*) *(see [6])*.

An f_0-p.c.a.a.b. $f : \xi' \to \xi$ defines an algebra pseudo-morphism $f^* : \Gamma(\xi') \to \Gamma(\xi)$, by $f^*(s') = f \circ s' \circ f_0$.

Definition 6 If for an f_0-p.c.a.a.b. $f : \xi' \to \xi$ there is an f_0-morphism of vector bundles $g : \xi \to \xi'$ which is a left inverse for f_x in every point $x \in M$, i.e. $g_x \circ f_x = id_{E_x}$, then we say that f is an f_0-*pseudo-comorphism of a.a.b.* which *splits* (f_0-*p.c.m.a.a.b.s.*). If f splits only locally, then we say that f is an f_0-*pseudo-comorphism of a.a.b* . which *splits locally* (f_0-*c.m.a.a.b.s.l.*). If additionally, $g_x(f_x(u) \cdot v) = u \cdot g_x(v)$, $g_x(v \cdot f_x(u)) = g_x(v) \cdot u$ and $g(\alpha \cdot v) = g(v)$ $(\forall) x \in M$, $u \in E_x$, $v \in E'_{f_0(x)}$, where $\alpha \in \Gamma(Z(\xi))$ is the central section of the comorphism f, then we say that the splitting g is regularly or f splits regularly .

We say that an f_0-p.c.m.f.a.a $f : \xi' \to \xi$ is regularly if it enjoys the property that $(\forall)\ X \in \mathcal{D}er(\Gamma(\xi))$, $s' \in \Gamma(\xi')$, $x \in M$, $(\exists) a \in E$ *so that* $X(f \circ s' \circ f_0)(x) = f(a)$.

Considering algebra comorphisms instead of pseudo-algebra comorphism (i.e. $\alpha = 1$), we obtain the corresponding notions for algebras.

Note that if f_0 is the identity, the comorphisms are also called morphisms.

We give now two examples.

1) For $\mathcal{A}' = \mathcal{A} = I\!R$ then $f : I\!R \to I\!R$, $f(x) = -x$ is an algebra pseudo-morphism. It splits by itself (i.e. $g = f$), but the splitting is not regularly.

2) Let $\xi = (E, \pi, M)$ be an a.a.b.. We suppose that the base M is compact, though the construction can be made, without major complications, for any manifold M. Let $\{U_i\}_{i=\overline{1,p}}$ be an open cover of the base M, such that there are the local trivialisations $\varphi_i; U_i \times \mathcal{A} \to \pi^{-1}(U_i)$. We denote as $t_{i,x} : \mathcal{A} \to \pi^{-1}(x)$, $t_{i,x}(a) = \varphi_i(x, a)$. Let $\{u_i\}_{i=\overline{1,p}}$ be a partition of unity on M subordinated to the open cover $\{U_i\}_{i=\overline{1,p}}$, such that *supp* $u_i \subset U_i$. Consider also the Uryson functions $\{v_i\}_{i=\overline{1,p}}$ according to the inclusions *supp* $u_i \subset U_i$. It is clear that $u_i v_i = u_i$, $(\forall) i = \overline{1,p}$. Let μ be the trivial a.a.b. which have the total space $M \times \mathcal{A}$, the base M and the model the algebra $\mathcal{A}$, and $\mu^p = \underbrace{\mu \oplus \cdots \oplus \mu}_{p\ ori}$ be the a.a.b. which is the Whitney sum and has as model the direct sum algebra $\underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{p\ ori}$.

The map $f : \xi \to \mu^p$ defined by $f(z) = (v_i(\pi(z)) \cdot t^{-1}_{i,\pi(z)}(z))_{i=\overline{1,p}}$ is a morphism and also a pseudo-morphism of a.a.b. (Since the vector bundles ξ and μ^p have the same base, the notions of morphism and comorphism are the same.). Notice that $\alpha = (v_i \cdot e)_{i=\overline{1,p}} \in Z(\Gamma(\xi))$. As it can be immediately seen, the (pseudo)-morphism f splits regularly by $g : \mu^p \to \xi$ defined by $g(x, (a_i)_{i=\overline{1,p}}) = \sum\limits_{i=1}^{p} u_i(x) \cdot t_{i.x}(a_i)$.

We present in that follows the results that can be extended from morphisms to pseudo-morphisms of a.a.b., as they are stated in [6]. The proofs are analogous and we omit them.

Proposition 2 *Let $\xi' = (E', \pi', M')$ and $\xi = (E, \pi, M)$ be two a.a.b., which have as models the algebras $\mathcal{A}'$ and $\mathcal{A}$ respectively, $f_0 : M \to M'$ be a map and $f : \xi' \to \xi$ be an f_0-p.c.m.a.a.b. which splits by the f_0-morphism of vector bundles $g : \xi \to \xi'$.*

If the splitting g is regularly or the pseudo-comorphism f is regularly, then the formula

$$\tau_{x_0} g(D)([s']) = g \circ D\left([f \circ s' \circ f_0]\right) - g(D([\alpha]))s'(x_0) \tag{1}$$

defines an application $\tau_{x_0} g : D^1(x_0, \xi) \to D^1(f(x_0), \xi')$.

Let $\xi' = (E', \pi', M')$ and $\xi = (E, \pi, M)$ be two a.a.b., which have as models the real algebras $\mathcal{A}'$ and $\mathcal{A}$ respectively, $f_0 : M \to M'$ be a map, $f : \xi' \to \xi$ be an f_0-pseudo-comorphism of a.a.b. which splits by the f_0-pseudo-morphism $g : \xi \to \xi'$. Then:

The map $\varphi : \Gamma(\xi') \to \Gamma(\xi)$, defined by $\varphi(s') = f \circ s \circ f_0$, is an algebra pseudo-morphism.

An f_0-morphism of vector bundles $\tau g : D^1(\xi) \to D^1(\xi')$ can be induced. This induces to its turn an $\mathcal{F}(M')$-module morphism

$$(\tau g)^* : \Gamma(D^1(\xi)) \to \mathcal{F}(M) \otimes_{\mathcal{F}(M')} \Gamma(D^1(\xi)),$$

where the tensor product is considered according to the real commutative algebras $f_0^* : \mathcal{F}(M') \to \mathcal{F}(M)$, $f_0^*(u') = u' \circ f_0$.

Theorem 1 *If $\dot{f} : \xi' \to \xi$ is a regularly f_0 -pseudo-comorphism of a.a.b., which splits by $\dot{g} : \xi \to \xi'$ then:*

1. *The map $\varphi : \Gamma(\xi') \to \Gamma(\xi)$, $\varphi(s') = g \circ s' \circ f_0$, is a pseudo-morphism of real algebras.*

2. *The map τg^* enjoys the properties:*

 a) *If $\tau g^*(X) = \sum\limits_i f_i \otimes_{\mathcal{F}(M')} X'_i$ and $s' \in \Gamma(\xi')$ then:*

$$[X, \varphi(s')] - [X, \alpha] \cdot \varphi(s') = \sum_i f_i \varphi\left([X'_i, s']\right). \tag{2}$$

 b) *Let X be as above and $\tau g^*(Y) = \sum\limits_\alpha g_\alpha \otimes_{\mathcal{F}(M')} Y'_\alpha$, then:*

$$\tau g^*\left([X, Y]\right) = \sum_{i,\alpha} f_i g_\alpha \otimes [X'_i, Y'_\alpha] + \sum_\alpha [X, g_\alpha] \otimes Y'_\alpha - \sum_i [Y, f_i] \otimes X'_i\ , \tag{3}$$

 where the tensor product is $\otimes_{\mathcal{F}(M')}$.

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University of Craiova,
Faculty of Mathematics and Informatics,
13, "Al.I.Cuza" Street, 1100 Craiova, ROMANIA
E-mail: paul@udjmath2.sfos.ro