On Transformation Groups of N-Linear Connections in the Bundle of Accelerations

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Abstract

In the present paper we define the notion of transformation of N-linear connections in the bundle of accelerations (Osc^2M, π, M), we prove that the set $\mathcal{T}$ of the transformations of N-linear connections on $E = Osc^2M$ together with the mapping composition is, in general, a non comutative group. In the group $\mathcal{T}$ there are some subgroups which keep invariant some of the components of an N- linear connection $D\Gamma(N)$ and $\mathcal{T}$ is a semidirect product of these subgroups. We study the transformation laws of the torsion and curvature tensor fields with respect to the transformations of the subgroup $\mathcal{T}_N$, that is the transformations of N-linear connections on E, which preserve the nonlinear connection N.

1 Preliminaries

Higher order Lagrange geometry had witnessed a wide acknowledgment due to the papers [5] – [9], published by R. Miron and Gh. Atanasiu.

The study of higher order Lagrange spaces is based on the k-osculator bundle notion. The bundle space of accelerations correspond in this study to $k = 2$, [1], [5].

Concerning the terminology and notations we use those from [10], which are essentially based on M. Matsumoto's book [2].

2 The group of transformations of N-linear connections

Let M be a C^∞-real differeniable, n-dimensional manifold and $\boldsymbol{(Osc^2M, \pi, M)}$ its **2 -osculator bundle**, or the bundle of accelerations, with $E = Osc^2M$ the total space.

The local coordinates on $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E, with the coefficients $N_{(1)}{}^i{}_j$, $N_{(2)}{}^i{}_j$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an N-linear connection on E.

If $\overline{N}$ is another nonlinear connection on E with the coefficients $\overline{N}_{(1)}{}^i{}_j(x, y^{(1)}, y^{(2)})$, $\overline{N}_{(2)}{}^i{}_j(x, y^{(1)}, y^{(2)})$, then there exist a uniquely determined tensor fields $A_{(\alpha)}{}^i{}_j \in \tau^1_1(E)$, $(\alpha = 1, 2)$ such that:

(2.1) $$\overline{N}_{(\alpha)}{}^{i}_{j} = N_{(\alpha)}{}^{i}_{j} - A_{(\alpha)}{}^{i}_{j}, \quad (\alpha = 1, 2).$$

Conversely, if $N_{(\alpha)}{}^{i}_{j}$ and $A_{(\alpha)}{}^{i}_{j}$, $(\alpha = 1, 2)$ are given, then $\overline{N}_{(\alpha)}{}^{i}_{j}$, $(\alpha = 1, 2)$ given by (2.1) is a nonlinear connection.

Let us consider an N-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$ and let us suppose that the mapping $N \to \overline{N}$ is given by (2.1).

According to Cap.III, §3.3, [11], we have:

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^{(\alpha)j}} = L^i_{jk} \frac{\delta}{\delta y^{(\alpha)i}} \quad , \quad D_{\frac{\delta}{\delta y^{(\beta)k}}} \frac{\delta}{\delta y^{(\alpha)j}} = C_{(\beta)}{}^{i}_{jk} \frac{\delta}{\delta y^{(\alpha)i}} \ ,$$

$$(\beta = 1, 2;\ \alpha = 0, 1, 2;\ y^{(0)i} = x^i).$$

It follows first of all that the transformations (2.1) preserve the coefficients $C_{(\alpha)}{}^{i}_{jk}$, $(\alpha = 1, 2)$. Taking in account the fact that:

$$\frac{\overline{\delta}}{\delta x^k} = \frac{\delta}{\delta x^k} + A_{(1)}{}^{h}_{k} \frac{\delta}{\delta y^{(1)h}} + A_{(2)}{}^{h}_{k} \frac{\delta}{\delta y^{(2)h}} \ ,$$

it follows:

$$D_{\frac{\overline{\delta}}{\delta x^k}} \frac{\delta}{\delta y^{(\alpha)j}} = \overline{L}^i_{jk} \frac{\delta}{\delta y^{(\alpha)i}} = D_{\frac{\delta}{\delta x^k} + A_{(1)}{}^{h}_{k} \frac{\delta}{\delta y^{(1)h}} + A_{(2)}{}^{h}_{k} \frac{\delta}{\delta y^{(2)h}}} \frac{\delta}{\delta y^{(\alpha)j}} =$$

$$= D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^{(\alpha)j}} + A_{(1)}{}^{h}_{k} D_{\frac{\delta}{\delta y^{(1)h}}} \frac{\delta}{\delta y^{(\alpha)j}} + A_{(2)}{}^{h}_{k} D_{\frac{\delta}{\delta y^{(2)h}}} \frac{\delta}{\delta y^{(\alpha)j}} =$$

$$= (L^i_{jk} + A_{(1)}{}^{h}_{k} C_{(1)}{}^{i}_{jh} + A_{(2)}{}^{h}_{k} C_{(2)}{}^{i}_{jh}) \frac{\delta}{\delta y^{(\alpha)i}}.$$

Therefore the change we are looking for is:

(2.2) $$\begin{cases} \overline{L}^i_{jk} = L^i_{jk} + A_{(1)}{}^{h}_{k} C_{(1)}{}^{i}_{jh} + A_{(2)}{}^{h}_{k} C_{(2)}{}^{i}_{jh}, \\ \overline{C}_{(\alpha)}{}^{i}_{jk} = C_{(\alpha)}{}^{i}_{jk}, \quad (\alpha = 1, 2). \end{cases}$$

So, we have proved:

Proposition 2.1 *The transformations (2.1) of nonlinear connections imply the transformations (2.2) for the coefficients* $L^i_{jk}, C_{(\alpha)}{}^{i}_{jk}$ $(\alpha = 1, 2)$ *of the* N*-linear connection* $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$.

Now, we can prove:

Theorem 2.1 *If* $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$ *and* $D\overline{\Gamma}(\overline{N}) = \overline{L}^i_{jk}, \overline{C}_{(1)}{}^{i}_{jk}, \overline{C}_{(2)}{}^{i}_{jk})$ *are two* N*-linear connections on the differentiable manifold* $E = Osc^2 M$*, then there exists only one quintet of tensor fields* $(A_{(1)}{}^{h}_{k}, A_{(2)}{}^{h}_{k}, B^i_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk})$ *such that:*

$$
(2.3)\qquad \begin{cases} \overline{N}_{(\alpha)}{}^{i}_{j} = N_{(\alpha)}{}^{i}_{j} - A_{(\alpha)}{}^{i}_{j}, \quad (\alpha = 1,2), \\ \overline{L}^{i}_{jk} = L^{i}_{jk} + A_{(1)}{}^{h}_{k} C_{(1)}{}^{i}_{jh} + A_{(2)}{}^{h}_{k} C_{(2)}{}^{i}_{jh} - B^{i}_{jk}, \\ \overline{C}_{(\alpha)}{}^{i}_{jk} = C_{(\alpha)}{}^{i}_{jk} - D_{(\alpha)}{}^{i}_{jk}, \quad (\alpha = 1,2). \end{cases}
$$

Proof: The first equality (2.3) determines uniquely the tensor fields $A_{(\alpha)}{}^{i}_{j}, (\alpha = 1,2)$, (see the exemple 1,§4, Cap I, [3] for Finsler spaces). Since $C_{(\alpha)}{}^{i}_{j}$, $(\alpha = 1,2)$ are tensor fields, the second equation (2.3) determines uniquely the tensor field B^{i}_{jk}. Similary the third equation (2.3) determines the tensor fields $D_{(\alpha)}{}^{i}_{j}$, $(\alpha = 1,2)$.

Theorem 2.2 *If $D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$ is an N-linear connection on E and $(A_{(1)}{}^{i}_{j}, A_{(2)}{}^{i}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk})$ is a quintet of tensor fields, then*

$$D\overline{\Gamma}(\overline{N}) = (\overline{L}^{i}_{jk}, \overline{C}_{(1)}{}^{i}_{jk}, \overline{C}_{(2)}{}^{i}_{jk})$$

given by (2.3) is an $\overline{N}$ -linear connection.

Proof: Indeed $\overline{N}_{(\alpha)}{}^{i}_{j} = N_{(\alpha)}{}^{i}_{j} - A_{(\alpha)}{}^{i}_{j}$, $(\alpha = 1,2)$ is a nonlinear connection. Since $A_{(1)}{}^{h}_{k} C_{(1)}{}^{i}_{jh} + A_{(2)}{}^{h}_{k} C_{(2)}{}^{i}_{jh} - B^{i}_{jk}$ and $D_{(\alpha)}{}^{i}_{jk}$, $(\alpha = 1,2)$ are tensor fields of the type (1,2), if follows that $\overline{L}^{i}_{jk}$ and $\overline{C}_{(\alpha)}{}^{i}_{jk}$, $(\alpha = 1,2)$ are the coefficients of an $\overline{N}$-linear connection $D\overline{\Gamma}(\overline{N}) = (\overline{L}^{i}_{jk}, \overline{C}_{(1)}{}^{i}_{jk}, \overline{C}_{(2)}{}^{i}_{jk})$.

The tensor fields $(A_{(1)}{}^{i}_{j}, A_{(2)}{}^{i}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk})$ are called the difference tensor fields of $D\Gamma(N)$ to $D\overline{\Gamma}(\overline{N})$ and the mapping $D\Gamma(N) \to D\overline{\Gamma}(\overline{N})$ given by (2.3) is called a transformation of N-linear connection to $\overline{N}$-linear connection.

Theorem 2.3 *The set $\mathcal{T}$ of the transformations of N-linear connections and composition of mappings is a group. The group $\mathcal{T}$ acts effectively and transitively on the set of N-linear connections.*

Proof: Let

$$(2.4)\quad t(A_{(1)}{}^{i}_{j}, A_{(2)}{}^{i}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk}) : D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk}) \to$$

$$\to D\overline{\Gamma}(\overline{N}) = (\overline{L}^{i}_{jk}, \overline{C}_{(1)}{}^{i}_{jk}, \overline{C}_{(2)}{}^{i}_{jk})$$

be a transformation of $\mathcal{T}$ given by (2.3). The composition of two transformations of $\mathcal{T}$ is a transformation of $\mathcal{T}$ given by

$$t(\overline{A}_{(1)}{}^{i}_{j}, \overline{A}_{(2)}{}^{i}_{j}, \overline{B}^{i}_{jk}, \overline{D}_{(1)}{}^{i}_{jk}, \overline{D}_{(2)}{}^{i}_{jk}) \circ t(A_{(1)}{}^{i}_{j}, A_{(2)}{}^{i}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk}) =$$

$$= t(A_{(1)}{}^{i}_{j} + \overline{A}_{(1)}{}^{i}_{j}, A_{(2)}{}^{i}_{j} + \overline{A}_{(2)}{}^{i}_{j}, B^{i}_{jk} + \overline{B}^{i}_{jk} + D_{(1)}{}^{i}_{jm} \overline{A}_{(1)}{}^{m}_{k} + D_{(2)}{}^{i}_{jm} \overline{A}_{(2)}{}^{m}_{k},$$

$$D_{(1)}{}^{i}_{jk} + \overline{D}_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk} + \overline{D}_{(2)}{}^{i}_{jk}).$$

The inverse of the transformation (2.4), is the transformation

$$t(-A_{(1)}{}^{i}{}_{j}, -A_{(2)}{}^{i}{}_{j}, -B^{i}_{jk} + D_{(1)}{}^{i}_{jm} A_{(1)}{}^{m}{}_{k} + D_{(2)}{}^{i}_{jm} A_{(2)}{}^{m}{}_{k}, -D_{(1)}{}^{i}_{jk}, -D_{(2)}{}^{i}_{jk}) :$$

$$: D\overline{\Gamma}(\overline{N}) = (\overline{L}^{i}_{jk}, \overline{C}_{(1)}{}^{i}_{jk}, \overline{C}_{(2)}{}^{i}_{jk}) \to D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk}).$$

The transformation $t(A_{(1)}{}^{i}{}_{j}, A_{(2)}{}^{i}{}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk})$ preserved all the connections $D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$ if $A_{(1)}{}^{i}{}_{j} = A_{(2)}{}^{i}{}_{j} = B^{i}_{jk} = 0$, $D_{(1)}{}^{i}_{jk} = D_{(2)}{}^{i}_{jk} = 0$. Therefore $\mathcal{T}$ acts effectively on the set of N-linear connections. From the Theorem 2.1 results that $\mathcal{T}$ acts transitively on this set q.e.d.

We introduce the following notations:

$$\mathcal{T}_{N} = \{t(0, 0, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk} \in \mathcal{T})\},\ \mathcal{T}_{\Gamma} = \{t(A_{(1)}{}^{i}{}_{j}, A_{(2)}{}^{i}{}_{j}, 0, 0, 0) \in \mathcal{T}\},$$

$$\mathcal{T}_{C} = \{t(A_{(1)}{}^{i}{}_{j}, A_{(2)}{}^{i}{}_{j}, B^{i}_{jk}, 0, 0) \in \mathcal{T})\},\ \mathcal{T}_{NC} = \{t(0, 0, B^{i}_{jk}, 0, 0) \in \mathcal{T})\},$$

$$\mathcal{T}_{NL} = \{t(0, 0, 0, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk} \in \mathcal{T})\}.$$

Proposition 2.2 *$\mathcal{T}_N, \mathcal{T}_\Gamma, \mathcal{T}_C, \mathcal{T}_{NC}$ and $\mathcal{T}_{NL}$ are Abelian subgroups of $\mathcal{T}$.*

Proposition 2.3 *The group $\mathcal{T}_\Gamma$ preserves the components $L^{i}_{jk}, C_{(1)}{}^{i}_{jk}$ and $C_{(2)}{}^{i}_{jk}$ of the N-linear connection $D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$; $\mathcal{T}_N$ preserves the nonlinear connection N, $\mathcal{T}_{NL}$ preserves $N_{(1)}{}^{i}{}_{j}, N_{(2)}{}^{i}{}_{j}, L^{i}_{jk}, \mathcal{T}_C$ preserves $C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk}$ and $\mathcal{T}_{NC}$ preserves $N_{(1)}{}^{i}{}_{j}, N_{(2)}{}^{i}{}_{j}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk}$.*

Proposition 2.4 *$\mathcal{T}_\Gamma, \mathcal{T}_N$ and $\mathcal{T}_{NC}$ are normal subgroups of $\mathcal{T}$.*

The proofs of these propositions are elementary. We can also notice that:
$t(A_{(1)}{}^{i}{}_{j}, A_{(2)}{}^{i}{}_{j}, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk}) = t(0, 0, B^{i}_{jk}, D_{(1)}{}^{i}_{jk}, D_{(2)}{}^{i}_{jk}) \circ t(A_{(1)}{}^{i}{}_{j}, A_{(2)}{}^{i}{}_{j}, 0, 0, 0)$.
But this product is not commutative. It results:

Theorem 2.4 *The group $\mathcal{T}$ is the semidirect product of the subgroup $\mathcal{T}_\Gamma$ and $\mathcal{T}_N : \mathcal{T} = \mathcal{T}_\Gamma \dot{\times} \mathcal{T}_N$, where $\dot{\times}$ denotes the semidirect product.*

But, every element from $\mathcal{T}$ is the product of some from $\mathcal{T}_C$ and $\mathcal{T}_{NL}$, too. Therefore it follows:

Theorem 2.5 *The group $\mathcal{T}$ is the semidirect product of the subgroup $\mathcal{T}_C$ and $\mathcal{T}_{NL}$: $\mathcal{T} = \mathcal{T}_C \dot{\times} \mathcal{T}_{NL}$.*

Thirdly, we can also notice:
$\mathcal{T}_N = \mathcal{T}_{NC} \times \mathcal{T}_{NL}$; $\mathcal{T}_C = \mathcal{T}_\Gamma \times \mathcal{T}_{NC}$, where $\times$ denotes the direct product.

From all these it follows:

Theorem 2.6 *The group $\mathcal{T}$ of the transformations of N-linear connections can be expressed as:* $\mathcal{T} = \mathcal{T}_\Gamma \dot{\times} (\mathcal{T}_{NC} \times \mathcal{T}_{NL}) = (\mathcal{T}_\Gamma \times \mathcal{T}_{NC}) \dot{\times} \mathcal{T}_{NL}$.

In short we have the following commutative diagram:

$$\begin{array}{ccc}
(N_{(1)}{}^i{}_j, N_{(2)}{}^i{}_j, L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk}) & \xrightarrow{\alpha} & (\overline{N}_{(1)}{}^i{}_j, \overline{N}_{(2)}{}^i{}_j, \overset{*}{L}{}^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk}) \xrightarrow{\gamma} \\
\Big\downarrow \beta & & \Big\downarrow \lambda \\
(N_{(1)}{}^i{}_j, N_{(2)}{}^i{}_j, \overset{**}{L}{}^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk}) & \xrightarrow{\alpha} & (\overline{N}_{(1)}{}^i{}_j, \overline{N}_{(2)}{}^i{}_j, \overline{L}^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk}) \xrightarrow{\gamma}
\end{array}$$

$$\begin{array}{c}
\xrightarrow{\gamma} (\overline{N}_{(1)}{}^i{}_j, \overline{N}_{(2)}{}^i{}_j, \overset{*}{L}{}^i_{jk}, \overline{C}_{(1)}{}^i_{jk}, \overline{C}_{(2)}{}^i_{jk}) \\
\Big\downarrow \mu \\
\xrightarrow{\gamma} (\overline{N}_{(1)}{}^i{}_j, \overline{N}_{(2)}{}^i{}_j, \overline{L}^i_{jk}, \overline{C}_{(1)}{}^i_{jk}, \overline{C}_{(2)}{}^i_{jk})
\end{array}$$

where

$\lambda = t(A_{(1)}{}^i{}_j, A_{(2)}{}^i{}_j, B^i_{jk}, 0, 0)$, $\mu = t(0, 0, B^i_{jk}, D_{(1)}{}^i_{jk}, D_{(2)}{}^i_{jk})$,
$\alpha = t(A_{(1)}{}^i{}_j, A_{(2)}{}^i{}_j, 0, 0, 0)$, $\beta = t(0, 0, B^i_{jk}, 0, 0)$, $\gamma = t(0, 0, 0, D_{(1)}{}^i_{jk}, D_{(2)}{}^i_{jk})$.

3 The transformations of the torsion and curvature tensor fields in the Abelian group $(\mathcal{T})_N$

Next we shall study the Abelian grup $\mathcal{T}_N$.
Its elements are the transformations $t(0, 0, B^i_{jk}, D_{(1)}{}^i_{jk}, D_{(2)}{}^i_{jk}) : D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk}) \to D\overline{\Gamma}(N) = (\overline{L}^i_{jk}, \overline{C}_{(1)}{}^i_{jk}, \overline{C}_{(2)}{}^i_{jk})$ given by

$$(3.1) \quad \overline{N}_{(\alpha)}{}^i{}_j = N_{(\alpha)}{}^i_j,\ \overline{L}^i_{jk} = L^i_{jk} - B^i_{jk},\ \overline{C}_{(\alpha)}{}^i_{jk} = C_{(\alpha)}{}^i_{jk} - D_{(\alpha)}{}^i_{jk},\ (\alpha = 1, 2).$$

Proposition 3.1 *The transformations of the Abelian group $\mathcal{T}_N$, given by (3.1) lead to the transformation of the torsions and curvatures in the following way:*

$$(3.2) \quad \overline{R}_{(0\alpha)}{}^i{}_{jk} = R_{(0\alpha)}{}^i{}_{jk}, \quad (\alpha = 1, 2),$$

$$(3.3) \quad \overline{T}_{(0)}{}^i{}_{jk} = T_{(0)}{}^i{}_{jk} + (B^i_{kj} - B^i_{jk}),$$

$$(3.4) \quad \overline{S}_{(\alpha)}{}^i{}_{jk} = S_{(\alpha)}{}^i{}_{jk} + (D_{(\alpha)}{}^i_{kj} - D_{(\alpha)}{}^i_{jk}), \quad (\alpha = 1, 2),$$

$$(3.5) \quad \overline{Q}_{(11)}{}^i{}_{jk} = Q_{(11)}{}^i{}_{jk} + (D_{(1)}{}^i_{kj} - D_{(1)}{}^i_{jk}),$$

$$(3.6) \quad \overline{Q}_{(12)}{}^i{}_{jk} = Q_{(12)}{}^i{}_{jk} - D_{(2)}{}^i_{jk},$$

$$(3.7)\quad \overline{Q}_{(21)}{}^{i}{}_{jk} = Q_{(21)}{}^{i}{}_{jk},$$

$$(3.8)\quad \overline{Q}_{(22)}{}^{i}{}_{jk} = Q_{(22)}{}^{i}{}_{jk} + D_{(1)}{}^{i}_{kj},$$

$$(3.9)\quad \overline{P}_{(\alpha\alpha)}{}^{i}{}_{jk} = P_{(\alpha\alpha)}{}^{i}{}_{jk} + B^{i}_{kj}, \quad (\alpha = 1,2),$$

$$(3.10)\quad \overline{P}_{(12)}{}^{i}{}_{jk} = P_{(12)}{}^{i}{}_{jk},$$

$$(3.11)\quad \overline{P}_{(21)}{}^{i}{}_{jk} = P_{(21)}{}^{i}{}_{jk},$$

$$(3.12)\quad \overline{R}_{h}{}^{i}{}_{jk} = R_{h}{}^{i}{}_{jk} - D_{(1)}{}^{i}_{hs}R_{(01)}{}^{s}{}_{jk} - D_{(2)}{}^{i}_{hs}R_{(02)}{}^{s}{}_{jk} - B^{i}_{hs}T_{(0)}{}^{s}{}_{jk} +$$
$$+\mathcal{A}_{jk}\{-B^{i}_{hj|k} + B^{s}_{hj}B^{i}_{sk}\},$$

$$(3.13)\quad \overline{P}_{(\alpha)h}{}^{i}{}_{jk} = P_{(\alpha)h}{}^{i}{}_{jk} - D_{(1)}{}^{i}_{hs}P_{(\alpha 1)}{}^{s}{}_{jk} - D_{(2)}{}^{i}_{hs}P_{(\alpha 2)}{}^{s}{}_{jk} - B^{i}_{hs}C_{(\alpha)}{}^{s}_{jk} -$$
$$-B^{i}_{hj}\;|^{(\alpha)}_{k} + D_{(\alpha)}{}^{i}_{hk|j} + B^{s}_{hj}D_{(\alpha)}{}^{i}_{sk} - D_{(\alpha)}{}^{s}_{hk}B^{i}_{sj},\ (\alpha = 1,2),$$

$$(3.14)\quad \overline{S}_{(21)h}{}^{i}{}_{jk} = S_{(21)h}{}^{i}{}_{jk} - C_{(2)}{}^{s}_{jk}D_{(1)}{}^{i}_{hs} + C_{(1)}{}^{s}_{kj}D_{(2)}{}^{i}_{hs} - D_{(1)}{}^{i}_{hj}\;|^{(2)}_{k} +$$
$$+D_{(2)}{}^{i}_{hk}\;|^{(1)}_{j} + D_{(1)}{}^{s}_{hj}D_{(2)}{}^{i}_{sk} - D_{(2)}{}^{s}_{hk}D_{(1)}{}^{i}_{sj} - D_{(2)}{}^{i}_{hs}B_{(21)}{}^{s}{}_{jk},$$

$$(3.15)\quad \overline{S}_{(\alpha\alpha)h}{}^{i}{}_{jk} = S_{(\alpha\alpha)h}{}^{i}{}_{jk} - D_{(\alpha)}{}^{i}_{hs}S_{(\alpha)}{}^{s}{}_{jk} + \mathcal{A}_{jk}\{-D_{(\alpha)}{}^{i}_{hj}\;|^{(\alpha)}_{k} +$$
$$+D_{(\alpha)}{}^{s}_{hj}D_{(\alpha)}{}^{i}_{sk}\} - D_{(\alpha)}{}^{i}_{hs}R_{(\alpha 2)}{}^{s}{}_{jk}, \quad (\alpha = 1,2),$$

where $\mathcal{A}_{ij}$ denotes the alternate summation.

We shall consider the tensor fields:

$$(3.16)\quad K_{h}{}^{i}{}_{jk} = R_{h}{}^{i}{}_{jk} - C_{(1)}{}^{i}_{hs}R_{(01)}{}^{s}{}_{jk} - C_{(2)}{}^{i}_{hs}R_{(02)}{}^{s}{}_{jk},$$

$$(3.17)\quad \mathcal{P}_{(1)h}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{P_{(1)h}{}^{i}{}_{jk} - C_{(1)}{}^{i}_{hs}\frac{\delta N_{(1)}{}^{s}{}_{j}}{\delta y^{(1)k}} - C_{(2)}{}^{i}_{hs}(N_{(1)}{}^{s}{}_{m}\frac{\delta N_{(1)}{}^{m}{}_{j}}{\delta y^{(1)k}} +$$
$$+\frac{\delta N_{(2)}{}^{s}{}_{j}}{\delta y^{(1)k}} - \frac{\delta N_{(2)}{}^{s}{}_{k}}{\delta y^{(1)j}})\},$$

$$(3.18)\qquad \mathcal{P}_{(2)h}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{P_{(2)h}{}^{i}{}_{jk} - C_{(1)}{}^{i}_{hs}\frac{\partial N_{(1)}{}^{s}{}_{j}}{\delta y^{(2)k}} - C_{(2)}{}^{i}_{hs}(N_{(1)}{}^{s}{}_{m}\frac{\partial N_{(1)}{}^{m}{}_{j}}{\delta y^{(2)k}} - -\frac{\partial N_{(2)}{}^{s}{}_{j}}{\partial y^{(2)k}})\}.$$

Proposition 3.2 *In a transformation (3.1) the tensor fields* $K_{h}{}^{i}{}_{jk}$, $\mathcal{P}_{(1)h}{}^{i}{}_{jk}$ *and* $\mathcal{P}_{(2)h}{}^{i}{}_{jk}$ *are transformed according to the following laws:*

$$(3.19)\qquad \overline{K}_{h}{}^{i}{}_{jk} = K_{h}{}^{i}{}_{jk} - B^{i}_{hs}T_{(0)}{}^{s}{}_{jk} + \mathcal{A}_{jk}\{-B^{i}_{hj|k} + B^{s}_{hj}B^{i}_{sk}\},$$

$$(3.20)\qquad \overline{P}_{(\alpha)h}{}^{i}{}_{jk} = P_{(\alpha)h}{}^{i}{}_{jk} - D_{(\alpha)}{}^{i}_{hs}T_{(0)}{}^{s}{}_{jk} - B^{i}_{hs}S_{(\alpha)}{}^{s}{}_{jk} + \mathcal{A}_{jk}\{-B^{i}_{hj}\,|^{(\alpha)}_{k} - D_{(\alpha)}{}^{i}_{hj|k} + +B^{s}_{hj}D_{(\alpha)}{}^{i}_{sk} + D_{(\alpha)}{}^{s}_{hj}B^{i}_{sk},\ (\alpha = 1, 2).$$

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