Common Supporting Lines for Families of Convex Bodies in the Plane

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Abstract

In this paper we give without proofs a survey of Helly-type theorems on the existence of a common supporting line for a family of convex bodies in Euclidean plane.

1 Introduction

Dawson and Edelstein (see [1] and [2]) have proved several Helly-type theorems on the existence of a common supporting hyperplane for a separated family of convex bodies in Euclidean space. They also mentioned that similar results could be obtained for the case of families of convex bodies that have restrictions on their positions less stringent than separation or disjointness.

Their methods of proofs are based on the considerations of some set-systems and on further investigations whether these systems can be realized geometrically. In what follows, we develop this method and complete some results from [2] on set-systems. We also improve the results by Dawson and Edelstein on common supporting lines for disjoint families of convex bodies in the plane and extend them for the case of a non-overlapping family, and even for a family with a given maximum number of members having non-empty-intersection (see [3]-[5]).

2 Definitions and notations

In what follows, we mean by a *planar convex body* any compact convex set in the plane, with interior points. A line l supports a planar convex body B provided l contains at least one boundary point of B and the whole B lies in a closed half-plane determined by l.

Following [1] and [2], we say that a family $\mathcal{F}$ of convex bodies in the plane has the (support) property S if there is a line l supporting every member of $\mathcal{F}$. If, additionally, all the members of $\mathcal{F}$ are in the same closed half-plane determined by l, we say that l is a $definite$ supporting line of the family and that $\mathcal{F}$ has property D. Similarly, $\mathcal{F}$ has property $S(n)$ (respectively, $D(n)$) if any subfamily of n members in $\mathcal{F}$ has property S (respectively, D). Trivially, $D \Rightarrow S$, $S \Rightarrow S(n)$ and $D \Rightarrow D(n)$; one has $S(n) \Rightarrow S(m)$ and $D(n) \Rightarrow D(m)$ for any $n \geq m$.

3 Transversality for set-systems, disjoint families of convex bodies in the plane

The results from [2] can be summarized as follows. If $\mathcal{F}$ is a finite family of n (pairwise) disjoint convex bodies in the plane, then:

1) $S(5) \Rightarrow S$, 2) $S(4) \Rightarrow S$ if $n \geq 7$, 3) $S(3) \Rightarrow S$ if $n \geq 237$.

As it was mentioned above, the proofs of these assertions are based on some results about set-systems as well as on further investigations on their geometrical realization. Our purpose here is to give a development of this method that leads to the completion of some known results. In particular (see [3]), the number 237 in assertion 3) above can be lowered to 143.

Let $\mathcal{A}$ be a set (finite or infinite) consisting of distinct elements, called by us letters. A set-system $\mathcal{L}$ on $\mathcal{A}$ consists of words, each being a subset of $\mathcal{A}$.

Our geometrical interpretation of a set-system is based on a family $\mathcal{F}$ of disjoint convex bodies in the plane, corresponding to distinct letters of the set-system. In this way a subfamily of $\mathcal{F}$ forms a word if and only if there is a line supporting all bodies of the subfamily such that no other body of $\mathcal{F}$ is supported by the line.

As it is shown in [2], any family of disjoint convex bodies in the plane has the following properties:

(1) any two bodies have exactly four common supporting lines,

(2) any three bodies have at most three common supporting lines,

(3) any five bodies have at most two common supporting lines.

Basing on these, we introduce two definitions.

Definition 1 A set-system $\mathcal{L}$ on the set $\mathcal{A}$ is called *special* provided it has the following properties:

(P1) no two letters in $\mathcal{A}$ are contained in more than four words,

(P2) no three letters in $\mathcal{A}$ are contained in more than three words,

(P3) no five letters in $\mathcal{A}$ are contained in more than two words.

Definition 2 A set-system $\mathcal{L}$ on the set $\mathcal{A}$ has the (transversal) property T if all the letters of $\mathcal{A}$ belong to a word. $\mathcal{A}$ has property $T(k)$, where k is a given positive integer, if any k (pairwise) distinct letters in $\mathcal{A}$ belong to a word.

In our interpretation, the property T (respectively, $T(k)$) of set-systems corresponds to the property S (respectively, $S(k)$) of families of disjoint convex bodies in the plane. Now we are able to formulate the main results from [3].

Theorem 1 $T(6) \Rightarrow T$ *for any special set-system.*

Theorem 2 $T(5) \Rightarrow T$ *for any special set-system on at least 7 letters.*

Theorem 3 $T(4) \Rightarrow T$ *for any special set-system on at least 11 letters.*

Theorem 4 $T(3) \Rightarrow T$ *for any special set-system on at least 143 letters.*

As a consequence of Theorem 4 we obtain

Corollary *$S(3) \Rightarrow S$ for any collection of at least 143 disjoint convex bodies in the plane.*

Since assertion 2) at the beginning of this section was shown to be tight (see [2]), the following problem still remains open.

Problem 1 What is the minimum positive integer n such that $S(3) \Rightarrow S$ for any family of at least n disjoint convex bodies in the plane?

Paper [3] contains an example of 16 disjoint convex bodies in the plane with the property $S(3)$ but not the property S. Hence the number n in Problem 1 is between 17 and 143.

In the same paper it is shown that assertions of Theorems 2 and 3 are tight, while the minimum number of letters in a special set-system $\mathcal{A}$, which guarantees the implication $T(3) \Rightarrow T$, is shown to be at least 21.

4 Non-overlapping families of convex bodies in the plane

A family of convex bodies is called $non-overlapping$ provided any two of the bodies do not have common interior points. Concerning definite common supporting lines, paper [4] contains the following result.

Theorem 5 *For any non-overlapping family of convex bodies in the plane, one has $D(4) \Rightarrow D$.*

Generally speaking, $D(3)$ does not imply D (see [4]), and an assertion similar to that of Theorem 5 does not hold for the property S. Moreover, for any positive integer n (≥ 3), there exists a non-overlapping family of at least n triangles with the property $S(n-1)$ but not the property S. Nevertheless, under an additional restriction, the following assertion can be proved (see [4]).

Theorem 6 *For any non-overlapping family of convex bodies in the plane, with no point in common, one has $S(6) \Rightarrow S$.*

An example from [4] gives an arbitrarily large non-overlapping family of triangles in the plane, having no point in common, with the property $S(5)$ but not the property S.

5 k-disjoint families of convex bodies in the plane

We will say that a family $\mathcal{F}$ of convex bodies is k-disjoint, where k (≥ 2) is a given positive integer, if no k of the bodies have common point. In particular, a 2-disjoint family means a disjoint family.

Theorem 7 *For any non-overlapping, k-disjoint family of at least 12k-13 convex bodies in the plane, one has $S(5) \Rightarrow S$.*

Theorem 8 *For any non-overlapping, k-disjoint family of at least 72k-75 convex bodies in the plane, one has $S(4) \Rightarrow S$.*

Paper [4] contains an example that shows that even a 3-disjoint family can have the property $D(3)$ (and hence the property $S(3)$) but not the property S. It is also given in [4] a non-overlapping, k-disjoint ($k \geq 5$) family of $k+2$ triangles with the property $S(5)$ but not the property S. In this connection, the following propblem can be stated.

Problem 2 To determine the least positive integer n, as a function of k, such that $S(5) \Rightarrow S$ (respectively, $S(4) \Rightarrow S$) for any non-overlapping, k-disjoint family of at least n convex bodies in the plane.

The proofs of the above results are based on the following property of k-disjoint families and on some extentions of the supporting properties considered in [1].

Property *Any 3k-1 non-overlapping, k-disjoint convex bodies in the plane have in common at most two supporting lines.*

In [5] we consider similar problems for the case of deffinite common supports. In particular, the following results are proved.

Theorem 9 *Let $\mathcal{F}$ be a k-disjoint family of cardinality n. Then:*

$$D(6) \Rightarrow D \text{ for } k \leq 7,$$
$$D(k-1) \Rightarrow D \text{ for } k \geq 7,$$
$$D(i) \Rightarrow D,\ 6 \leq i \leq k-2, \text{ for } k \geq 8 \text{ and } n \geq k.$$

The last assertion of this theorem is tight for any positive integer $k \geq 8$ (see [5]). One can ask about the implication $D(i) \Rightarrow D$ for $i < 6$ and the following problem appears first.

Problem 3 Let k (≥ 3) be a given positive integer. Does there exist a positive integer $n = n(k)$ such that $D(5) \Rightarrow D$ for any k-disjoint family of at least n members?

The results below give a positive answer to this question for the cases $k = 3$ and $k = 4$.

Theorem 10 $D(5) \Rightarrow D$ *for any 3-disjoint family.*

Theorem 11 $D(5) \Rightarrow D$ *for any 4-disjoint family of at least seven bodies.*

References

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