Estimates of the Heat Content on a Class of Domains

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Abstract

We give sharp upper and lower bounds of the heat content of a domain in a Riemannian manifold. These bounds hold for all values of time, under the assumption that the Ricci curvature of the domain and the mean curvature of its boundary are both non-negative. These estimates extend to convex sets of the Euclidean space having arbitrary boundary; in that case, we also give an upper bound of the trace of the heat kernel.

1 Introduction

Let Ω be a domain in a complete Riemannian manifold (M, g) of dimension m.

We assume $\bar{\Omega}$ compact and $\partial\Omega$ piecewise-smooth. Let $u_t = u(t, \cdot)$ be the solution of the heat equation on Ω which satisfies the Dirichlet condition on the boundary ($u_t = 0$ on $\partial\Omega$) and which has initial data $u_0 = \mathbf{1}_\Omega$. Denote by $H(t)$ the *total heat content of* Ω *at time* t :

$$H(t) = \int_\Omega u_t(x)\, dx, \tag{1}$$

where dx is the Riemannian measure. Some of the previous results concerning this function have focused mainly on its asymptotic behavior for small times. We first mention the following result:

$$H(t) = vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t} + \frac{m-1}{2}\int_{\partial\Omega} \eta \cdot t + O(t^{3/2}). \tag{2}$$

as $t \to 0$, and where η is the mean curvature. This holds for a domain with *smooth* boundary in any Riemannian manifold and has first been proved by van den Berg and LeGall in [3] for domains in Euclidean space $\mathbb{R}^n$. Keeping the assumption of smoothness of $\partial\Omega$ (things get rather complicated otherwise, see below) one can go

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on; and in fact van den Berg and Gilkey proved in [2] that there exists an asymptotic series, as $t \to 0$:

$$H(t) = vol(\Omega) + \sum_{k=1}^{\infty} \beta_k t^{k/2} \tag{3}$$

and proceeded to compute $\beta_1, \dots, \beta_4$ for Riemannian manifolds with boundary. The authors employ the functorial properties of the coefficients, which brings them to the remarkable generalization of the above calculation to operators of Laplace type on sections of a vector bundle. However, the method does not produce an estimate for the remainder term of (2).

It has been a preoccupation of the author to estimate this remainder term for domains in arbitrary Riemannian manifold (see [13]). This calculation is one of the consequences of a general method (see the announcement paper [11], to be followed by publication [12]). This method involves the distance function from a submanifold, and has proven to be effective in this kind of problems. In particular, the method produces a recursive algorithm for the calculation of the *whole* asymptotic series (3), which can be found in [10].

The scope of this exposition is to show some explicit estimates of the heat content, which hold for *all* times t, and not just in the asymptotic limit as $t \to 0$; the main preoccupation is to *measure* effectively the amount of heat held by the domain, and to *compare* it with the heat content of special domains (in our case, flat cylindrical surfaces). All the results below, with the exception of Theorem 4, will appear in [10], to which we refer for complete proofs.

The main estimate relates the heat content with the volume of the *parallel sets* $\Omega(r) = \{x \in \Omega : dist(x, \partial\Omega) > r\}$.

Theorem 1. *Let $\partial\Omega$ be smooth, and assume that both the Ricci curvature of Ω and the mean curvature of its boundary are non-negative. Then, for all $t > 0$:*

$$\frac{4}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/t} vol(\Omega(r))\, dr - vol(\Omega) \ \leq\ H(t) \ \leq\ \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/4t} vol(\Omega(r))\, dr.$$

Both bounds are *optimal* because, when Ω is a flat cylinder (that is, $\Omega = N \times (0, 2R)$ where N is a closed manifold and the metric is given by the product), then both the upper and the lower bound approach $vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t}$ as $R \to \infty$. In fact, for flat cylinders, $\mathrm{vol}(\Omega(r))$ is equal to $vol(\Omega) - r \cdot vol(\partial\Omega)$ if $r < R$, and is zero if $r > R$; so the gap between the upper and the lower bound is an exponentially decreasing function of t, which goes to zero when t is fixed and R tends to infinity.

Remark. The theorem continues to hold if $\partial\Omega$ is only piecewise-smooth, if one assumes in addition that the foot of any geodesic segment which minimizes the distance from the boundary is a regular point of the boundary. So the theorem holds for convex polyhedrons in $\mathbb{R}^m$, and by Hausdorff density it holds for *any* convex domain in $\mathbb{R}^m$.

Remark. Really the condition we need for the theorem to be true is that the Laplacian of the function which measures the distance from the boundary is a *positive* distribution (see section 2).

The main theorem has a number of consequences; we mention in this introduction the simplest ones, and refer to section 2 for sharper estimates, and especially for the estimates on convex sets.

Corollary 2. *Under the assumptions of Theorem 1, we have, for all $t > 0$:*

$$H(t) \geq vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t}.$$

The Corollary follows from the main estimate and the fact that, for all r, one has $vol(\Omega(r)) \geq vol(\Omega) - r \cdot vol(\partial\Omega)$.

Corollary 3. *If $l_1(\Omega)$ is the first eigenvalue of the Dirichlet problem of Ω, then:*

$$l_1(\Omega) \leq \frac{4e^2}{\pi(e-1)^2}\left(\frac{vol(\partial\Omega)}{vol(\Omega)}\right)^2 .$$

This estimate on the first eigenvalue follows easily from Corollary 2 and the Fourier series representation of the heat content (given in (5) below). Regarding the third term in the expansion (2), we mention what happens if we drop the assumption of smoothness for $\partial\Omega$. Then the first three terms of the asymptotic expansion of the heat content have been computed in the following cases: by van den Berg and Srisatkunarajah (see [4]) for polygonal regions in the euclidean plane, and by the author (see [12]) for convex polyhedrons in the euclidean space of arbitrary dimension. The fact is that the term in t, which in the smooth case is a multiple of the integral mean curvature, does not pass to the limit under smooth approximations of the boundary, and thus some other methods have to be considered.

It should perhaps be emphasized, however, that our distance function method applies also for domains with piecewise-smooth boundaries, and a calculation similar to that done for convex polyhedrons should give the general expression of the third term of the expansion in this general case (at least when the smooth pieces of the boundary intersect transversally).

The heat content and the trace of the heat kernel. Let $k(t,x,y)$ be the Dirichlet heat kernel of Ω (i.e. the fundamental solution of the heat equation on Ω with Dirichlet boundary conditions). The spectrum of Ω is completely determined by the integral:

$$Z(t) = \int_{\Omega} k(t,x,x)\,dx = \sum_{k=1}^{\infty} e^{-l_k t} , \tag{4}$$

where $\{l_k\}$ is the sequence of the Dirichlet eigenvalues of the Laplace operator Δ (each being counted with multiplicity). Now let $\{\phi_k\}$ be a corresponding orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions. Then the heat content admits the following Fourier series representation:

$$H(t) = \sum_{k=1}^{\infty} \left\{ \int_{\Omega} \phi_k \right\}^2 e^{-l_k t} . \tag{5}$$

Regarding the function $Z(t)$, our estimates for the volume of $\Omega(r)$ (see Proposition 6) produces the following estimate when Ω is convex. In what follows, R is the *inner radius* of Ω and d is the constant: $d = inf_{x \in \rho^{-1}(R)} sup_{y \in \partial\Omega}\{d(x,y)\}$. Clearly $R \leq d < diam(\Omega)$, and $d = R$ iff Ω is a ball.

Theorem 4. *Assume that Ω is a convex subset of $\mathbb{R}^m$. Then, for all $t>0$:*

$$Z(t) \leq (4\pi t)^{-n/2}\left(vol(\Omega) - \frac{\sqrt{\pi}}{2} vol(\partial\Omega)\sqrt{t} + C_2 t\right)$$

where $C_2 = 2^{m-3}(m-1)vol(S^{m-1})\frac{d^{m-1}}{R}$.

We notice that the first two terms in the right-hand side coincide with the first two terms of the asymptotic expansion of $Z(t)$ as $t \to 0$ (see [9]).

Proof. Fix $x \in \Omega$. Then there exists a half-space, call it $H(x)$, which contains Ω and is bounded by an hyperplane at distance $\rho(x) = d(x, \partial\Omega)$ from x. As $k(t,x,x)$ is bounded above by the corresponding quantity on $H(x)$ (maximum principle), we have:

$$k(t,x,x) \leq \frac{1}{(4\pi t)^{n/2}}\left(1 - e^{-\rho(x)^2/t}\right).$$

Integrating over Ω, using the coarea formula (see [7]), and then integrating by parts, we obtain:

$$(4\pi t)^{n/2} Z(t) \leq \frac{2}{t}\int_0^\infty r e^{-r^2/t} \cdot \mathrm{vol}(\Omega(r))\, dr$$

We now apply the estimates of Proposition 6 and conclude. □

There is a vast literature on the trace of the heat kernel. It would be interesting to investigate the relation between the functions $Z(t)$ and $H(t)$; let us only observe from (5) that the heat content depends only on those eigenvalues in the eigenspace of which there is an eigenfunction not integrating to zero (for example, on an interval these eigenvalues make up exactly half of the spectrum).

There are many examples showing that the knowledge of the spectrum does not determine the isometry class of the manifold (this question was raised in the famous paper [8] and started the fascinating subject of isospectrality). In other words, non-isometric domains may produce identical "sound".

A question which arises naturally in the framework of this exposition is the following: *Does the spectrum determine the heat content at all times?*

2 The method

For $x \in \Omega$, denote by $\rho(x)$ the *minimum distance* of x from $\partial\Omega$. This defines the *distance function* $\rho : \Omega \to (0,\infty)$. One sees that ρ is *Lipschitz regular* , hence continuous on Ω, but definitely not smooth on Ω. More precisely, ρ is smooth on the set of its *regular points* $\Omega \setminus Cut(\partial\Omega)$, and the singular set $Cut(\partial\Omega)$ (which is the *cut-locus* of the normal exponential map) has always *zero-measure in* Ω (this fact is classical for the cut-locus of a point; see [13, App.D] for a detailed proof in our case).

An essential ingredient in our method is the *distributional Laplacian* $\Delta\rho$ *of the distance function* (we are forced to consider distributions because ρ is only Lipschitzian). First, fix a regular point x of Ω. As $\|\nabla\rho(x)\| = 1$, we see that the level set (parallel submanifold) $\rho^{-1}(\rho(x))$ is locally a *smooth submanifold* near x. A straightforward calculation shows that:

$\Delta\rho(x) =$ trace of the second fundamental form of the level set through x.

This is what we call the *regular part* $\Delta_{reg}\rho$ of $\Delta\rho$. To get the whole Laplacian, we have to add its *singular part* , which is a distribution $\Delta_{cut}\rho$ supported on the cut-locus. It is not difficult to show that $\Delta_{cut}\rho$ is *positive* , hence a measure on Ω. As $\Delta_{reg}\rho$ turns out to be in $L^1(\Omega)$ (see [13]), we conclude that $\Delta\rho$ is itself a *measure* on Ω.

We now state the fundamental technical lemma, to be extensively used in the sequel. This lemma holds more generally for the distance function to any piecewise-smooth submanifold of M (for its proof, see [13]). In what follows, $\Omega(r)$ will denote the set of points of Ω which are at distance greater than r from the boundary of Ω.

Mean-value lemma. ([11, Thm 1.1]) *Let* $u \in C^2(\Omega)$, *and for* $r > 0$, *let* $F(r) = \int_{\Omega(r)} u\, dv_n$. *Then the following equality holds as measures on* $(0, \infty)$*:*

$$F''(r) = -\int_{\Omega(r)} \Delta u + \rho_*(u\Delta\rho)(r) \ ,$$

where $\rho_*(u\Delta\rho)$ *is the push-forward measure of* $u\Delta\rho$ *by* ρ.

The push-forward is naturally defined by the formula: $\int_0^\infty \psi\rho_*(u\Delta\rho) = \int_\Omega u(\psi \circ \rho)\Delta\rho$. About the name, it is justified by the fact that the above formula easily implies the classical mean-value lemma for harmonic functions on $\mathbb{R}^n$.

The lemma above can be applied to any differential equation involving the Laplacian, transforming it in a differential equation depending only on a real variable (the distance from the boundary). In particular:

Applications to eigenvalue estimates: let u be an eigenfunction of the Laplace operator on M: $\Delta u = lu$. Then $F(r) = \int_{\Omega(r)} u$ satisfies:

$$F''(r) + lF(r) = -\rho_*(u\Delta\rho)(r) \ . \tag{6}$$

Applications to heat content estimates: Let $u(t, x)$ be a solution of the heat equation on M. Then $F(t, r) = \int_{\Omega(r)} u_t$ satisfies the following heat equation on $(0, \infty)$:

$$\left(-\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t}\right) F(t, r) = \rho_*(u(t, \cdot)\Delta\rho)(r) \ . \tag{7}$$

We refer to [12] for applications to eigenvalue estimates; in this exposition we consider equation (7) which will allows us to reduce the study of heat diffusion on Ω to a *one-dimensional* heat equation.

It is also clear how the bounds of the heat content will follow from the assumption that the measure $\Delta\rho \geq 0$, because then, as u_t is non-negative, we can apply the maximum principle to (7).

3. Some consequences of the main estimate

Geometric assumption. We assume from now on that the distributional Laplacian of the distance function is a *positive measure* on Ω. If $\partial\Omega$ is smooth, this is simply equivalent to require that the mean curvature of each level set of the distance function is non-negative (at its regular points). In general, we have the following:

Sufficient conditions. Let Ω be piecewise-smooth. If both the *mean curvature* of the regular part of $\partial\Omega$, and the *Ricci curvature* of Ω, are non-negative, and if the foot of any geodesic segment which minimizes the distance from the boundary is a regular point of the boundary, then $\Delta\rho \geq 0$.

For the proof, just observe that, under the curvature assumptions, by Bochner formula, the mean curvature is non-decreasing on any geodesic normal to the boundary, so $\Delta_{reg}\rho$, hence $\Delta\rho$, is a positive measure on Ω.

The main estimate takes the form:

Theorem 1. *Assume $\Delta\rho \geq 0$. Then, for all $t > 0$:*

$$\frac{4}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/t} vol(\Omega(r))\, dr - vol(\Omega) \ \leq \ H(t) \ \leq \ \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-r^2/4t} vol(\Omega(r))\, dr.$$

These estimates can be extended, by polyhedral approximation, to any bounded, convex subset of $\mathbb{R}^m$.

Geometric estimates involving the volume of the parallel domains $\Omega(r)$ will then give rise, using Theorem 1, to the desired estimates of the heat content.

We discuss two cases.

Case 1: $\partial\Omega$ is smooth. Let I_{inj} denote the injectivity radius of the normal exponential map. Then our function is smooth on $(0, Inj)$ and can be expanded around $r = 0$:

$$vol(\Omega(r)) = vol(\Omega) - vol(\partial\Omega) r + \frac{m-1}{2} \int_{\partial\Omega} \eta \cdot r^2 + R_2(r)$$

and $R_2(r)$, the remainder term, can be controlled by the curvatures of Ω. One gets:

Corollary 5. *Let $\partial\Omega$ be smooth, and fix $0 < a < R_{inj}$. Then, for all $t > 0$:*

$$vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t} + \frac{m-1}{2} \int_{\partial\Omega} \eta \cdot t + c_3 t^{3/2} - g(t) \leq H(t)$$

$$\leq vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t}(1 - e^{-a^2/4t}) + (m-1) \int_{\partial\Omega} \eta \cdot t + 4C_3 \cdot t^{3/2} \ ,$$

where η is the mean curvature of $\partial\Omega$,

$c_3 = min\{inf_{r\in(0,a)} C(r), 0\}$,

$C_3 = max\{sup_{r\in(0,a)} C(r), 0\}$,

$C(r) = \frac{1}{3\sqrt{\pi}} \int_{\rho^{-1}(r)} \left(scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)} \right) dv_{m-1}$

and $g(t) = \frac{m-1}{a\sqrt{\pi}}\left(\int_{\partial\Omega}\eta\right)(2a^2t^{1/2} + 2t^{3/2})e^{-\frac{a^2}{t}}$.

Observe that the first three terms of the lower bound coincide with those of the expansion (2). In other words, the lower bound of Corollary 5 is sharp up and including the term in t, as $t \to 0$.

Case 2: Ω is a convex subset of $\mathbb{R}^m$. We stress that we assume *no regularity at all* for the boundary of the convex set (which is not even piecewise-smooth, in general). The convexity property, however, will imply global estimates of the volume of the parallel domains, which we are going to show. First, we recall some classical invariants of a convex set.

Let B_r be the open ball of radius r centered at the origin of $\mathbb{R}^m$. Then $\Omega + B_r$ is the (exterior) parallel set which consists of all points of $\mathbb{R}^m$ whose distance from Ω is less than r. For all $r > 0$, the *Steiner-Minkowski formula* holds (see [5],[7]):

$$vol(\Omega + B_r) = \sum_{k=0}^{m} V_k(\Omega)r^k. \tag{8}$$

The numbers $V_k(\Omega)$ are called the *cross-sectional measures* of Ω; they are non-negative, and continuous with respect to the *Hausdorff metric* on convex sets. In particular: $V_0(\Omega) = \mathrm{vol}(\Omega)$, $V_1(\Omega) = vol(\partial\Omega)$, the *canonical* $(m-1)$*−volume of the boundary,* and $V_2(\Omega)$ is the *integral mean curvature.* This name is justified by the fact that, when $\partial\Omega$ is C^2−smooth, then $V_2(\Omega) = \frac{m-1}{2}\int_{\partial\Omega}\eta$, where η is the mean curvature.

We can now give the relevant bounds of $vol(\Omega(r))$. For this, we need two more invariants: the inner radius R, and the constant d which is defined in the Introduction.

Proposition 6. *Let Ω, d, R be as above, let $m \geq 2$, and let $vol(\partial\Omega) = V_1(\Omega)$ (resp. $V_2(\Omega)$) be the canonical $(m-1)$-volume (resp. the integral mean curvature) of $\partial\Omega$. Then, for all $r \geq 0$:*

$$vol(\Omega) - r\cdot vol(\partial\Omega) + V_2(\Omega)r^2 - c_3r^3 \leq vol(\Omega(r)) \leq$$

$$\leq vol(\Omega) - r\cdot vol(\partial\Omega) + \frac{d}{R}V_2(\Omega)r^2 + C_3r^3 \ ,$$

where, if $m = 2$: $V_2(\Omega) = \pi$, $C_2 = \frac{\pi d}{R}$, $c_3 = \frac{\pi}{R}$, and $C_3 = 0$;
if $m \geq 3$: $C_2 = 2^{m-3}(m-1)\mathrm{vol}(S^{m-1})\frac{d^{m-1}}{R}$; $c_3 = 2^{m-3}(m-1)(m-2)\mathrm{vol}(S^{m-1})\frac{d^{m-2}}{3R}$, and $C_3 = \frac{d}{2R}c_3$.

Substituting in Theorem 1, we get:

Corollary 7. *Let Ω be any bounded, open, convex subset of $\mathbb{R}^m$, $m \geq 2$. Then, for all $t > 0$:*

$$vol(\Omega) - \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + V_2(\Omega)\cdot t - \tilde{c}_3t^{3/2} \leq H(t) \leq$$

$$\leq vol(\Omega) - \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + \frac{2d}{R}V_2(\Omega)\cdot t + \tilde{C}_3t^{3/2},$$

where, if $m = 2:\ V_2(\Omega) = \pi$, $\widetilde{C}_2 = \frac{2\pi d}{R}$, $\widetilde{c}_3 = \frac{2\sqrt{\pi}}{R}$, $\widetilde{C}_3 = 0$*; and, if* $m \geq 3$*:* $\widetilde{C}_2 = 2^{m-2}(m-1)\cdot vol(S^{m-1})\frac{d^{m-1}}{R}$, $\widetilde{c}_3 = \frac{2^{m-2}}{3\sqrt{\pi}}(m-1)(m-2)vol(S^{m-1})\cdot\frac{d^{m-2}}{R}$, *and* $\widetilde{C}_3 = \frac{2d}{R}\widetilde{c}_3$.

Remark. If $\partial\Omega$ is smooth, the lower bound is again sharp up and including the third term, as $t \to 0$; however, this is no longer true if $\partial\Omega$ has corners, and in fact, if the boundary is *polyhedral,* the coefficient $\beta_2(1)$ of t in the asymptotic expansion of the heat content satisfies in that case the strict inequality: $\beta_2(1) > V_2(\Omega)$ (see [12], Theorem 3.3 for its calculation). For example, for the unit square, $\beta_2(1) = \frac{16}{\pi}$ while $V_2(\Omega) = \pi$ (which shows, by the way, that $\beta_2(1)$ does not pass to the limit under smooth approximations of the polyhedron).

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