Principal Directions for Submanifolds Imbeded in Euclidean Spaces of Arbitrary Codimension

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Abstract

In a recent paper [4] are introduced principal directions of a locally imbedded n-dimensional manifold in $\mathbb{R}^{n+k}$, which generally are not intrinsically defined. In this paper are introduced another two definitions of principal directions using sectional curvature and Ricci tensor, which are intrinsically defined. Some basic results about hypersurfaces are proven. Further, there are considered some special cases when some of the these principal directions coincide.

1 Introduction

In the paper [4] it is considered a locally embedded n-dimensional manifold M in the Euclidean space $\mathbb{R}^{n+k}$. Let $\{Y_i\}(1 \leq i \leq n)$ be an arbitrary orthonormal system of tangent vector fields on the manifold, and let $\{N_\alpha\}(1 \leq \alpha \leq k)$ be an arbitrary orthonormal system of vector fields orthogonal to the tangent space. Considering an arbitrary tensor field T on M, it can be prolonged locally in an $(n+k)$–dimensional neighbourhood such that

$$\nabla_N T \equiv 0, \tag{1.1}$$

for each orthogonal unit vector N, where ∇ denotes the covariant differentiation with respect to the Euclidean metric. Usually, by N we will denote a unit vector or a vector field which is orthogonal to the basic manifold. For an arbitrary vector field X, it implies

$$[N, X] = \nabla_N X - \nabla_X N = -\nabla_X N \tag{1.2}$$

and hence $N \perp [N, X]$ because $N \cdot [N, X] = 0$. It also holds $X \cdot [N, Y] = Y \cdot [N, X]$ for an arbitrary normal vector field N and arbitrary tangent vector fields X and Y.

In the paper [4] are introduced the principal directions and the so called principal normal vectors of the manifold M imbedded in $\mathbb{R}^{n+k}$, and also are introduced canonical k normal vector fields and the corresponding curvatures, which do not depend on the choice of the parameterization of M. Now we shall recall the definition of the principal directions and the principal normal vectors.

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The following tensor field T_{ij} is defined as follows

$$T_{ij} = \sum_{\alpha=1}^{k}\sum_{r=1}^{n}(Y_i \cdot [N_\alpha, Y_r])(Y_r \cdot [N_\alpha, Y_j]). \tag{1.3}$$

It is a symmetric tensor field and invariant of the orthonormal basis $Y_1, \cdots, Y_n$. If $(\lambda_1, \cdots, \lambda_n)$ is an arbitrary eigenvector of the matrix T_{ij}, then the vector $\lambda_1 Y_1 + \cdots + \lambda_n Y_n$ is an eigenvector for the corresponding (1.1) tensor field T^i_j, and it does not depend on the basis $Y_1, \cdots, Y_n$ or $N_1, \cdots, N_k$. The eigenvectors are orthogonal and they determine the *principal directions.* Their eigenvalues are non-negative real numbers.

Indeed these eigenvalues are squares of the "principal curvatures" which we would like to be defined. They are determined up to the sign. It is shown why it is not convenient to determine the n-tuple $(\lambda_1, \cdots, \lambda_n)$ of "principal curvatures" up to the sign, as it is done for the special case $k = 1$. So principal normal vectors are introduced as follows.

Suppose that $Z_1, \cdots, Z_n$ are orthonormal vectors which determine the principal directions. Then we define the corresponding *principal normal vectors* $U_i (1 \leq i \leq n)$ by

$$U_i = \sum_{\alpha=1}^{k}(Z_i \cdot [N_\alpha, Z_i])N_\alpha \tag{1.4}$$

and also define the vectors

$$U_{ij} = \sum_{\alpha=1}^{k}(Z_i \cdot [N_\alpha, Z_j])N_\alpha \tag{1.5}$$

such that $U_i = U_{ii}$. From the definition (1.3) it follows that these vectors are canonical. Indeed, it can be verified easily that they do not depend on the choice of the orthonormal system $\{N_\alpha\}$ and do not depend on the signs of the vectors Z_i. The following proposition holds.

Proposition 1.1 *The sums* $\sum_{j=1}^{n} || U_{ij} ||^2$ *for* $i \in \{1, \cdots, n\}$ *are the eigenvalues of* T.

Note that the matrix

$$\sum_{\alpha=1}^{k}\sum_{r=1}^{n}(Z_i \cdot [N_\alpha, Z_r])(Z_r \cdot [N_\alpha, Z_j]) \tag{1.6}$$

is diagonal and hence

$$\sum_{\alpha=1}^{k}\sum_{r=1}^{n}(Z_i \cdot [N_\alpha, Z_r])(Z_r \cdot [N_\alpha, Z_j]) = 0 \qquad (i \neq j). \tag{1.7}$$

In the special case $k = 1$ it is proven that the principal directions defined here coincide with the standard principal directions from the classical literature. It follows that then $U_{ij} = 0$ for $i \neq j$ and $\| U_i \|^2$ $(1 \leq i \leq n)$ are the squares of the principal curvatures. In this paper another two definitions for principal directions which are intrinsic are given, i.e. they depend only on the metric of the manifold. Further, there are considered some special cases when some of these definitions give the same principal directions.

2 Two intrinsic definitions of principal directions

The principal directions given in section 1 do not always depend only on the metric of the imbedded manifold, i.e. they are not always intrinsic. However here we give another two definitions of principal directions which are intrinsic.

First we consider the sectional curvature if $n > 2$. We are going to find the expression for the sectional curvature with the given notations. We denote by ∇ the covariant derivative in the Euclidean space $\mathbb{R}^{n+k}$, and thus

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \equiv 0. \tag{2.1}$$

The projection of ∇ on the base manifold induces a connection ∇' given by

$$\nabla'_X Z = \nabla_X Z - \sum_{\alpha=1}^{k} (N_\alpha \cdot \nabla_X Z) N_\alpha \tag{2.2}$$

for vector fields X and Z on the base manifold. The corresponding Riemannian curvature tensor yields to

$$R'(X,Y)Z = ([\nabla'_X, \nabla'_Y] - \nabla'_{[X,Y]})Z = R(X,Y)Z+$$

$$+\sum_{r=1}^{n}\sum_{\alpha=1}^{k}(Z\cdot\nabla_Y N_\alpha)(Y_r\cdot\nabla_X N_\alpha)Y_r - \sum_{r=1}^{n}\sum_{\alpha=1}^{k}(Z\cdot\nabla_X N_\alpha)(Y_r\cdot\nabla_Y N_\alpha)Y_r +$$

$$+\sum_{\alpha=1}^{k}(N_\alpha \cdot \nabla_{[X,Y]} Z) N_\alpha\ , \tag{2.3}$$

where Y_i is an arbitrary basis of the tangent space. Using that $R(X,Y)Z = 0$, the sectional curvature $K(Y_i, Y_j)$ of the plane generated by Y_i and Y_j is given by

$$K(Y_i,Y_j) = Y_i \cdot R'(Y_i,Y_j)Y_j =$$

$$= \sum_{\alpha=1}^{k}(Y_i\cdot[N_\alpha,Y_i])(Y_j\cdot[N_\alpha,Y_j]) - \sum_{\alpha=1}^{k}(Y_i\cdot[N_\alpha,Y_j])(Y_j\cdot[N_\alpha,Y_i]) =$$

$$= U_i \cdot U_j - U_{ij}\cdot U_{ij} \qquad (i \neq j)\ , \tag{2.4}$$

where

$$U_i = \sum_{\alpha=1}^{k}(Y_i\cdot[N_\alpha,Y_i])N_\alpha \text{ and } U_{ij} = \sum_{\alpha=1}^{k}(Y_i\cdot[N_\alpha,Y_j])N_\alpha.$$

Hence it holds

$$K(Y_i, Y_j) \leq U_i \cdot U_j. \tag{2.5}$$

Now we are going to define principal directions on the tangent bundle, using the sectional curvature. We call the directions of the orthonormal system of vectors $Z_1, \cdots, Z_n (n \geq 3)$ to be principal, if $\sum_{1 \leq i < j \leq n} (K(Y_i, Y_j))^2$ has a local maximum for $Y_i = Z_i (1 \leq i \leq n)$, considering all orthonormal systems $Y_1, \cdots, Y_n$. Since $O(n, r)$ is a compact manifold, local (global) maximum will exist, and such principal directions always exist, but they may not be unique. It is important that they are *intrinsic principal directions* of the manifold (also for arbitrary Riemannian manifold). The definition of these principal directions is not convenient for practical calculation. The following theorem gives necessary and sufficient conditions for the local maximum of $\sum_{1 \leq i < j \leq n} (K(Y_i, Y_j))^2$, for an arbitrary n-dimensional Riemannian manifold.

Theorem 2.1 *Let M be an arbitrary n-dimensional Riemannian manifold ($n > 2$). Then $\sum_{1 \leq i < j \leq n} (K(Y_i, Y_j))^2$ has a local maximum for $Y_i = Z_i (1 \leq i \leq n)$ if and only if*

$$A = \sum_{r \neq i,j} (K(Z_j, Z_r) - K(Z_i, Z_r)) R_{rirj} = 0 \tag{2.6}$$

and

$$B = \sum_{r \neq i,j} \Big((K(Z_j, Z_r) - K(Z_i, Z_r))^2 - 4(R_{rirj})^2 \Big) \geq 0 \tag{2.7}$$

for each $1 \leq i < j \leq n$, where $R_{srij} = R(Z_s, Z_r, Z_i, Z_j) = Z_s \cdot R(Z_i, Z_j) Z_r$.

Proof. Without loss of generality we can suppose that M is (locally) imbedded in $\mathbb{R}^{n+k}$. An infinitesimal rotation can be represented as a composition of rotations in the plane generated by Z_i and Z_j of different pairs $i < j$. For the sake of simplicity we will consider a rotation in the $Z_i Z_j$-plane of angle φ, and let

$$Z_i' = Z_i \cos \varphi + Z_j \sin \varphi, \qquad Z_j' = -Z_i \sin \varphi + Z_j \cos \varphi.$$

It is sufficient to prove that

$$0 \geq \sum_{r \neq i,j} \Big(K(Z_i', Z_r)^2 + K(Z_j', Z_r)^2 - K(Z_i, Z_r)^2 - K(Z_j, Z_r)^2 \Big). \tag{2.8}$$

Using the identity (2.4) one obtains

$$K(Z_i', Z_r) = K(Z_i, Z_r) \cos^2 \varphi + K(Z_j, Z_r) \sin^2 \varphi +$$

$$+ 2 \sin \varphi \cos \varphi \Big((Z_i \cdot [N_\alpha, Z_j])(Z_r \cdot [N_\alpha, Z_r]) - (Z_i \cdot [N_\alpha, Z_r])(Z_j \cdot [N_\alpha, Z_r]) \Big),$$

$$K(Z_j', Z_r) = K(Z_j, Z_r) \cos^2 \varphi + K(Z_i, Z_r) \sin^2 \varphi -$$

$$-2\sin\varphi\cos\varphi\Big((Z_i\cdot[N_\alpha,Z_j])(Z_r\cdot[N_\alpha,Z_r])-(Z_i\cdot[N_\alpha,Z_r])(Z_j\cdot[N_\alpha,Z_r])\Big).$$

Putting this in (2.8) and using from (2.3) that

$$R_{srij}=R(Z_s,Z_r,Z_i,Z_j)=Z_s\cdot R(Z_i,Z_j)Z_r=U_{rj}\cdot U_{si}-U_{ri}\cdot U_{sj}=$$

$$(2.9)\qquad =\sum_{\alpha=1}^{k}\Big((Z_r\cdot[N_\alpha,Z_j])(Z_s\cdot[N_\alpha,Z_i])-(Z_r\cdot[N_\alpha,Z_i])(Z_s\cdot[N_\alpha,Z_j])\Big),$$

the inequality (2.8) is equivalent to

$$0\geq -2B\sin^2\varphi\cos^2\varphi-4A\cos\varphi\sin\varphi(\cos^2\varphi-\sin^2\varphi),$$

where A and B are defined by (2.6) and (2.7) respectively. This inequality is satisfied for arbitrary φ, if and only if A=0 and $B\geq 0$.□

Note that another intrinsic principal directions can be defined by the maximum of the expression $\sum_{1\leq i<j\leq n}(K(Y_i,Y_j))^{2s}$, and hence they are not unique. Note that the conditions (2.6) and (2.7) are intrinsic.

Now we have the following important consequence.

Theorem 2.2 *If the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}(n>2)$ are different numbers, then the corresponding principal directions and the principal curvatures up to the sign depend only on the metric, i.e. they are intrinsic properties of the hypersurface.*

Proof. It is proven in [4] that if the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$ are different numbers, then $Z_i\cdot[N,Z_j]=0$ for $i\neq$j. Hence, A=0 and

$$B=\sum_{r\neq i,j}(K(Z_j,Z_r)-K(Z_i,Z_r))^2\geq 0.$$

Thus we have a local maximum for the expression $\sum_{1\leq i<j\leq n}(K(Y_i,Y_j))^2$ and so the principal directions are intrinsic properties.

Further, since $K(Z_i,Z_j)=\lambda_i\lambda_j\ (i\neq j)$, where $\lambda_1,\cdots,\lambda_n$ are the principal curvatures, they can be determined as follows:

$$\mid\lambda_i\mid=\Big(K(Z_i,Z_j)K(Z_j,Z_r)K(Z_r,Z_i)\Big)^{1/2}/\mid K(Z_j,Z_r)\mid\quad,\qquad K(Z_j,Z_r)\neq 0,$$

where j and r are chosen such that $i\neq j\neq r\neq i$. Let $\lambda_1\neq 0$, then $\lambda_i\neq 0(i\neq 1)$ has the same sign as λ_1 if and only if $K(Z_1,Z_i)>0$. Hence, the principal curvatures up to the sign are also determined by the metric. □

The Theorem 2.2 is a known result because under some conditions the second fundamental form up to the sign depends only on the intrinsic metric [3], but here we have given another proof.

Now we give another definition for the principal directions. Indeed, instead of the tensor T in (1.3) we can consider the following tensor

$$(2.10)\quad S_{ij}=\sum_{\alpha=1}^{k}\sum_{r=1}^{n}\Big((Y_i\cdot[N_\alpha,Y_r])(Y_r\cdot[N_\alpha,Y_j])-(Y_i\cdot[N_\alpha,Y_j])(Y_r\cdot[N_\alpha,Y_r])\Big).$$

All the properties for the tensor T are satisfied also by S and moreover S is an intrinsic tensor because according to (2.10), $S_{ij} = \sum_r R_{rijr}$ is the Ricci tensor. Note that for any principal directions determined by any definition, the corresponding principal normal vectors U_i and also the vectors U_{ij} are defined by (1.4) and (1.5).

3 Special cases

The principal directions defined in section 1 and those in section 2 do not always coincide. However there exist some special cases when they coincide. In this section we will consider such special cases.

First we note that the principal direction using the sectional curvature can be applied only for $n > 2$. Also the method using the Ricci tensor can be applied only for $n > 2$ because if $n = 2$, then the Ricci tensor has the form $S_{ij} = \lambda\delta_{ij}$ in a coordinate system where $g_{ij} = \delta_{ij}$ at the considered point. So, if $n = 2$, it can be applied only the definition given in section 1.

Now let us choose a point and a coordinate system such that $g_{ij} = \delta_{ij}$ at the considered point and $R_{rirj} = 0$ for $i \neq j \neq r \neq i$. We prove that in this case the definitions using sectional curvature and Ricci tensor give the same principal directions. Indeed, let $Z_1, \cdots, Z_n$ be the unit tangent vectors to the coordinates $x^1, \cdots, x^n$ respectively. We prove that the directions determined by $Z_1, \cdots, Z_n$ are principal directions for these two definitions. It is obvious that that (2.6) and (2.7) are satisfied. Further it is obvious that $S_{ij} = 0$ for $i \neq j$ because $R_{iiij} = R_{jijj} = 0$. Hence the proof is finished.

This special case often occurs for $n = 3$. Indeed, suppose that Z_1, Z_2 and Z_3 determine principal directions according to the definition using the sectional curvature, and suppose that $K(Z_1, Z_2), K(Z_2, Z_3)$ and $K(Z_3, Z_1)$, are different numbers. Then for given i and $j (i \neq j)$ there exists only one index r such that $i \neq j \neq r \neq i$, and hence (2.6) implies that $R_{rirj} = 0$ for $i \neq j \neq r \neq i$. Thus $S_{ij} = 0$ for $i \neq j$, and Z_1, Z_2 and Z_3 determine principal directions using the Ricci tensor.

Note that $R_{rirj} = 0$ for $i \neq j \neq r \neq i$ denotes that

$$U_{ir} \cdot U_{jr} - U_{ij} \cdot U_{rr} = 0, \qquad i \neq j \neq r \neq i. \tag{3.1}$$

Hence

$$\sum_{r=1}^{n} U_{ir} \cdot U_{rj} = \sum_{r\neq i,j}^{n} U_{ir} \cdot U_{rj} + U_{ii} \cdot U_{ij} + U_{ij} \cdot U_{jj} =$$

$$= \sum_{r\neq i,j}^{n} U_{ij} \cdot U_{rr} + U_{ii} \cdot U_{ij} + U_{ij} \cdot U_{jj} = U_{ij} \cdot \sum_{r=1}^{n} U_{rr}.$$

Thus in more special case, when $U_{ij} = 0$ for $i \neq j$, all three definitions give the same principal directions. Note that $U_{ij} = 0$ for $i \neq j$ implies (3.1), i.e. $R_{rirj} = 0$ for $i \neq j \neq r \neq i$. In this special case ($U_{ij} = 0$ for $i \neq j$) it holds

$$K(Z_i, Z_j) = U_i \cdot U_j \qquad (i \neq j), \tag{3.2}$$

and we will prove that $\| U_i \|^2 = U_i \cdot U_i$ depends only on the intrinsic metric, assuming that the manifold locally is isometrically imbedded in the Euclidean space $\mathbb{R}^{n+r}$ for the smallest possible dimension r.

Indeed, let $Z_1, \cdots, Z_n$ be the principal directions. We consider the following matrix

$$L_{ij} = K(Z_i, Z_j) = U_i \cdot U_j \qquad (i \neq j)$$

and require for the diagonal elements $L_{ii}(1 \leq i \leq n)$ such that the $n \times n$ matrix L has the smallest rank as possible. Its rank is $\leq k$. Indeed, the diagonal elements should be of the form $U_i \cdot U_i(1 \leq i \leq n)$. If $a_1, \cdots, a_{k+1}$ and $b_1, \cdots, b_{k+1}$ are vectors in $\mathbb{R}^k$, then

$$rank \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_{k+1} \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \cdots & a_2 \cdot b_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k+1} \cdot b_1 & a_{k+1} \cdot b_2 & \cdots & a_{k+1} \cdot b_{k+1} \end{bmatrix} \leq k$$

because this matrix is a sum of k matrices with rank ≤ 1. Hence

$$\begin{vmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_{k+1} \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \cdots & a_2 \cdot b_{k+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k+1} \cdot b_1 & a_{k+1} \cdot b_2 & \cdots & a_{k+1} \cdot b_{k+1} \end{vmatrix} = 0$$

and $rank(L) \leq k$. This number $rank(L)$ represents the smallest codimension of local isometric imbedding of the manifold in the Euclidean space. These elements $L_{ii}(1 \leq i \leq n)$ are unique determined and they are just the squares $\| U_i \|^2 = U_i \cdot U_i$. Thus they depend only on the metric. Indeed, here we determine a frame in $\mathbb{R}^r$ $(r = rank(L))$, which is unique up to the isometric transformation in $\mathbb{R}^r$.

Remark. We do not know anything when this statement can be generalized, for example if $U_{ij} \neq 0$ for $i \neq j$.

At the end we consider a more special case than the previous one, $k = 1 (n > 2)$. Then there exists a system such that $U_{ij} = 0$ for $i \neq j$ and all three definitions give the same principal directions as the classical one. Since the same principal directions gives the definition using the Ricci tensor, we obtain another proof of the Theorem 2.2.

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