Some Remarks on the Quantomorphism Group

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Abstract

If M is the quantizing manifold of N, the infinitesimal quantomorphism Lie algebra is isomorphic to $C^\infty(N,\mathbb{R})$ with the Poisson bracket and the universal covering of the identity component of the quantomorphism group is a direct Lie group product of the universal covering of the group of Hamiltonian diffeomorphisms on N with $\mathbb{R}$. All the groups involved are ILH Lie groups as well as regular convenient Lie groups (for the Frölicher-Kriegl differential calculus).

1 Introduction

A contact structure on a manifold M of dimension $2n+1$ is a maximally non-integrable distribution of codimension one, which is also transversally orientable. So $\xi = \operatorname{Ker}\alpha$ for a 1-form α on M such that $\alpha \wedge (d\alpha)^n \neq 0$. Given a contact form α, there exists a unique field X_α on M, called the Reeb vector field, such that $i(X_\alpha)d\alpha = 0$ and $i(X_\alpha)\alpha = 1$.

A diffeomorphism φ on M is called a contactomorphism if it preserves the oriented hyperplane field $\xi = \operatorname{Ker}\alpha$. This is equivalent to $\varphi^*\alpha = e^h\alpha$ for some function $h: M \to \mathbb{R}$. A vector field X on M which satisfies $\mathcal{L}_X\alpha = g\alpha$ for some function $g: M \to \mathbb{R}$ is called a contact vector field. X is a contact vector field if and only if there exists a function $f: M \to \mathbb{R}$ such that $i(X)\alpha = -f$ and $i(X)d\alpha = df - X_\alpha(f)\alpha$, so there is a one-to-one correspondence between contact vector fields and smooth functions.

The group of contact diffeomorphisms $\operatorname{Diff}(M,\xi)$ for compact M is an ILH Lie group [O] with Lie algebra $\mathcal{X}(M,\xi)$, the Lie algebra of contact vector fields. It is also a regular convenient Lie group (i.e. for the Frölicher-Kriegl differential calculus

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on convenient vector spaces [FK]) with the space of compactly supported contact vector fields as Lie algebra [KM].

The contact form α is called regular if the flow of the Reeb vector field defines a free circle action Φ on M. Then the quotient manifold $N = M/S^1$ is smooth and carries a symplectic structure ω, $[\omega] \in H^2(N, \mathbb{Z})$ and $\pi^*\omega = d\alpha$ where $\pi : M \to N$ is the canonical projection. Under these assumptions $\pi : M \to N$ is a principal S^1-bundle, α a principal connection one-form and ω its curvature two form. The Reeb vector field X_α generates the vertical bundle, the principal right action of S^1 on M is Φ and $\xi = \operatorname{Ker}\alpha$ is the horizontal bundle. M is called the quantizing manifold of N.

A diffeomorphism φ on M is called a quantomorphism if φ preserves the 1-form α. We denote by $\operatorname{Diff}(M, \alpha)$ the group of quantomorphisms. The infinitesimal quantomorphisms are the vector fields on M which satisfy $\mathcal{L}_X \alpha = 0$. Let $\mathcal{X}(M, \alpha)$ be the Lie algebra of infinitesimal quantomorphisms. Proposition 1 states that it is isomorphic to $C^\infty(N, \mathbb{R})$ with the Poisson bracket, the isomorphism being exactly the restriction of the isomorphism above between $\mathcal{X}(M, \xi)$ and $C^\infty(M, \mathbb{R})$.

Two different proofs that the quantomorphism group $\operatorname{Diff}(M, \alpha)$ for compact M is an ILH Lie group with Lie algebra $\mathcal{X}(M, \alpha)$ can be found in Omori [O] and Ratiu-Schmid [RS]. It is shown in Proposition 4 that $\operatorname{Diff}(M, \alpha)$ for compact M is also a regular convenient Lie group.

2 The Lie algebra of infinitesimal quantomorphisms

Let (M, α) be a compact quantizing manifold of (N, ω). Let $\mathcal{X}_{S^1}(M)$ be the Lie algebra of S^1-invariant vector fields on M. Then

$$0 \longrightarrow C^\infty(N, \mathbb{R}) \xrightarrow{J} \mathcal{X}_{S^1}(M) \xrightarrow{P} \mathcal{X}(N) \longrightarrow 0. \tag{1}$$

is an exact sequence of Lie algebras, where on $C^\infty(N, \mathbb{R})$ we consider the trivial bracket. Here $J(g) = (g \circ \pi)X_\alpha$ and $P(X) = Y$ is the projection on N of the invariant (hence projectable) vector field X. Every $X \in \mathcal{X}_{S^1}(M)$ decomposes into its horizontal and vertical parts $X = CY - (g \circ \pi)X_\alpha$ where $g \in C^\infty(N, \mathbb{R})$ and CY is the horizontal lift of $Y = P(X)$. Indeed the vector field $X = CY + fX_\alpha \in \mathcal{X}_{S^1}(M)$ $\Leftrightarrow [X, X_\alpha] = 0 \Leftrightarrow [CY, X_\alpha] = X_\alpha(f)X_\alpha \Leftrightarrow X_\alpha(f) = 0 \Leftrightarrow f = -g \circ \pi$. From this decomposition the exactness of the sequence follows immediately.

The restriction of the exact sequence (1) to the Lie algebra of infinitesimal quantomorphisms gives a central extension of Lie algebras (i.e. the image of J is contained in the center of $\mathcal{X}(M, \alpha)$):

Proposition 1. *The central extension of Lie algebras*

$$0 \longrightarrow \mathbb{R} \xrightarrow{J} \mathcal{X}(M, \alpha) \xrightarrow{P} \mathcal{X}_{Ham}(N, \omega) \longrightarrow 0. \tag{2}$$

is trivial. The Lie algebra of infinitesimal quantomorphisms of (M, α) is isomorphic to $C^\infty(N, \mathbb{R})$ with the Poisson bracket.

Proof. All infinitesimal quantomorphisms have the form $X = CH_g - (g \circ \pi)X_\alpha$ for some $g \in C^\infty(N, \mathbb{R})$ because in the decomposition $X = CY - (g \circ \pi)X_\alpha$ we have the equivalences $\mathcal{L}_X \alpha = 0 \Leftrightarrow i_{CY} d\alpha = d(g \circ \pi) \Leftrightarrow \pi^* i_Y \omega = \pi^* dg \Leftrightarrow Y = H_g$.

We define a splitting of (2) by $S : \mathcal{X}_{Ham}(N,\omega) \to \mathcal{X}(M,\alpha)$, $S(Y) := CH_g - (g \circ \pi)X_\alpha$, where $g \in C^\infty(N,\mathbb{R})$ is the unique Hamiltonian with zero integral $\int_N g\omega^n = 0$ of the Hamiltonian vector field $Y = H_g$. Then S is really a Lie algebra homomorphism:

$$\begin{aligned}
[S(H_{g_1}), S(H_{g_2})] &= [CH_{g_1}, CH_{g_2}] + (CH_{g_2}(g_1 \circ \pi)X_\alpha - CH_{g_1}(g_2 \circ \pi)X_\alpha \\
&= CH_{\{g_1,g_2\}} + ((-\omega(H_{g_1}, H_{g_2}) + H_{g_2}(g_1) - H_{g_1}(g_2)) \circ \pi)X_\alpha \\
&= CH_{\{g_1,g_2\}} - (\{g_1, g_2\} \circ \pi)X_\alpha = S([H_{g_1}, H_{g_2}]).
\end{aligned}$$

Here the second step follows from the fact that ω is the curvature form of the principal connection α on M, the third step uses the definition of the Poisson bracket $\{g_1, g_2\} = -\omega(H_{g_1}, H_{g_2}) = H_{g_1}(g_2) = -H_{g_2}(g_1)$, and the forth step relies on the fact that every Poisson bracket has integral zero $\int_N \{g_1, g_2\}\omega^n = 0$.

It follows that the Lie algebra of infinitesimal quantomorphisms of (M, α) is isomorphic to the direct product of the Lie algebra of Hamiltonian vector fields with $\mathbb{R}$, hence it is isomorphic to $C^\infty(N,\mathbb{R})$ with the Poisson bracket. □

The isomorphism of Lie algebras constructed in the Proposition above,

$$\begin{aligned}
A : (C^\infty(N,\mathbb{R}), \{,\}) &\to (\mathcal{X}(M,\alpha), [,]) \ , \\
A(g) &= CH_g - (g \circ \pi)X_\alpha \ ,
\end{aligned}$$

fits into the following commutative diagram of Lie algebras:

$$\begin{array}{ccc}
\mathcal{X}(M,\alpha) & \xrightarrow{A^{-1}} & C^\infty(N,\mathbb{R}) \\
i \downarrow & & \downarrow \pi^* \\
\mathcal{X}(M,\xi) & \xrightarrow{-\alpha} & C^\infty(M,\mathbb{R}) \ .
\end{array}$$

Here we consider on $C^\infty(M,\mathbb{R})$ the bracket induced from $\mathcal{X}(M,\xi)$ by $-\alpha$:

$$\{f_1, f_2\}_\alpha = X_1(f_2) + X_\alpha(f_1)f_2 \ ,$$

where X_i is the unique vector field in $\mathcal{X}(M,\xi)$ such that $-\alpha(X_i) = f_i$. We have to verify only that π^* becomes a Lie algebra morphism:

$$\{g_1\circ\pi, g_2\circ\pi\}_\alpha = X_1(g_2\circ\pi) = (CH_{g_1} - (g_1\circ\pi)X_\alpha)(g_2\circ\pi) = H_{g_1}(g_2)\circ\pi = \{g_1, g_2\}\circ\pi.$$

3 Exact sequences of regular convenient Lie groups

In this paragraph we show there are exact sequences of regular convenient Lie groups which integrate the exact sequences (1) and (2) of Lie algebras. The convenient smooth manifolds are defined by gluing C^∞-open sets in convenient vector spaces via smooth diffeomorphisms (for the Frölicher-Kriegl differential calculus [FK]. A locally convex vector space is said to be convenient if any Mackey-Cauchy-sequence converges. The C^∞-topology of a locally convex vector space E is the final topology with respect to all smooth curves into E. A mapping between locally convex vector spaces is called smooth if it takes smooth curves into smooth curves. A convenient Lie group is a convenient smooth manifold and a group such that the group operations are smooth. The Lie group G with Lie algebra $\mathfrak{g}$ is called regular if every smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ can be integrated to a smooth curve $g \in C^\infty(\mathbb{R}, G)$ with $g(0) = e$ and the evolution map $evol : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ defined by $evol(X) = g(1)$ is smooth. For diffeomorphism groups the evolution operator is just integration of time dependent vector fields with compact support. The reference for regular convenient Lie groups is the book [KM].

An exact sequence of Lie groups integrating (1) is

$$0 \longrightarrow C^\infty(N, S^1) \xrightarrow{j} \mathrm{Diff}_{S^1}(M) \xrightarrow{p} \mathrm{Diff}(N) \longrightarrow 0. \tag{1'}$$

Here $j(s) = \Phi_s$ and $p(\varphi) = \psi$ where $\pi \circ \varphi = \psi \circ \pi$.

$C^\infty(N, S^1)$ is the gauge group of the principal bundle M over N. It is an ILH Lie group and also a regular convenient Lie group [KM].

Let G be a compact Lie group acting smoothly on M. The group of G-equivariant diffeomorphisms on M is

$$\mathrm{Diff}_G(M) = \{\varphi \in \mathrm{Diff}(M) : \varphi(gx) = g\varphi(x), \forall g \in G\}$$

and the Lie algebra of G-invariant vector fields on M is

$$\mathcal{X}_{c,G}(M) = \{X \in \mathcal{X}_c(M) : g^*X = X, \forall g \in G\}.$$

For compact M the group $\mathrm{Diff}_G(M)$ is an ILH Lie group with Lie algebra $\mathcal{X}_G(M)$ [EM]. The next proposition shows it is also a regular convenient Lie group.

Proposition 2. $\mathrm{Diff}_G(M)$ *is a closed regular convenient Lie subgroup of* $\mathrm{Diff}(M)$ *with Lie algebra* $\mathcal{X}_{c,G}(M)$*, the space of compactly supported* G*-invariant vector fields on* M*.*

Proof. Because G is compact we can find G-invariant Riemannian metrics on M. Let $\exp : TM \supseteq U \to M$ be the associated exponential map defined on the open neighborhood U_0 of the zero section such that $(\pi_M, \exp) : U_0 \to V \subset M \times M$ is a smooth diffeomorphism.

Let U be the open neighborhood of id in $\mathrm{Diff}(M)$ of those compactly supported diffeomorphisms whose graph is contained in V and let $u : U \to \mathcal{X}_c(M)$ be the map $u(\varphi)(x) = \exp^{-1}(\varphi(x))$. The inverse of u is $u^{-1}(X)(x) = \exp_x(X(x))$.

We prove the pair (U, u) is a submanifold chart for $\mathrm{Diff}_G(M)$. This means to prove the equivalence: $X \in \mathcal{X}_{c,G}(M)$ if and only if $\varphi := u^{-1}(X) \in \mathrm{Diff}_G(M)$. The following equivalences for $X = u(\varphi)$ end the proof that (U, u) is a submanifold chart:

$$\begin{aligned} g^*X = X &\Leftrightarrow T_x g.X(x) = X(g(x)), \forall x \in M \\ &\Leftrightarrow g(\exp X(x)) = \exp(X(gx)), \forall x \in M \\ &\Leftrightarrow g\varphi(x) = \varphi(gx), \forall x \in M. \end{aligned}$$

Here the second step follows from the fact that isometries move geodesics to geodesics.

A submanifold chart in the neighborhood of an arbitrary $\psi \in \mathrm{Diff}_G(M)$ is obtained by translating the chart (U, u). Hence $\mathrm{Diff}_G(M)$ is a closed submanifold of $\mathrm{Diff}(M)$ and a Lie group because composition and inversion are smooth (by restriction). It is also regular because it is a group of diffeomorphisms. □

Proposition 3. [RS] *An exact sequence of ILH Lie groups corresponding to (2) is*

$$0 \longrightarrow S^1 \xrightarrow{j} \mathrm{Diff}(M,\alpha) \xrightarrow{p} \mathcal{K} \longrightarrow 0.$$

with $j(s) = \Phi_s$ and $p(\varphi) = \psi$ for $\pi \circ \varphi = \psi \circ \pi$. Here $\mathcal{K}$, the group of those symplectomorphisms of (N,ω) that can be lifted to quantomorphisms of (M,α), equals the group of those symplectomorphisms ψ of (N,ω) for which both parallel transports along a piecewise smooth curve c and along its image $\psi\circ c$ are restrictions of the same Φ_s, $s \in S^1$. The quantomorphism group is also a principal S^1-fiber bundle over $\mathcal{K}$.

Taking the components of the identity we get the central extension

$$0 \longrightarrow S^1 \xrightarrow{j} \mathrm{Diff}(M,\alpha)_0 \xrightarrow{p} \mathrm{Ham}(N,\omega) \longrightarrow 0 \tag{2‘}$$

where $\mathrm{Ham}(N,\omega)$ is the group of Hamiltonian diffeomorphisms of (N,ω). Recall that a symplectomorphism ψ is called Hamiltonian if there is an isotopy ψ_t from $\psi_0 = id$ to $\psi_1 = \psi$ such that $\dot{\psi}_t \in \mathrm{Ham}(N,\omega)$ where

$$\frac{d}{dt}\psi_t = \dot{\psi}_t \circ \psi_t.$$

In [KM] it is shown that $\mathrm{Ham}(N,\omega)$ is a regular convenient Lie group.

Proposition 4. *Let M be a compact quantizing manifold. Then $\mathrm{Diff}(M,\alpha)_0$, the identity component in the group of quantomorphisms, is a regular convenient Lie group with Lie algebra $\mathcal{X}(M,\alpha)$.*

Proof. First we define a principal bundle atlas. Every local section $h : W \to M$ of the bundle $\pi : M \to N$ with W an open subset of N, together with a pair of points $x_0 \in M$, $z_0 \in N$ such that $\pi(x_0) = z_0$, define a local section σ of (2) and so a principal bundle chart for $\mathrm{Diff}(M,\alpha)_0$ like follows:

$$\sigma : U = \{\psi \in \mathrm{Ham}(N,\omega) : \psi(z_0) \in W\} \to \mathrm{Diff}(M,\alpha)_0,$$

$\sigma(\psi)$ is defined to be that lift φ of ψ whose value at x_0 is $\varphi(x_0) = h(\psi(z_0))$. This definition is correct because all lifts of ψ differ by some $\Phi_s, s \in S^1$ and all quantomorphisms commute with the circle action Φ. The principal bundle chart is $\Psi : U \times S^1 \to \mathrm{Diff}(M,\alpha)_0$, $\Psi(\psi,s) = \sigma(\psi)\circ\Phi_s$. The compatibility of the charts: Let Ψ_1, Ψ_2 bundle charts defined by sections h_1, h_2 and points $(x_1,z_1), (x_2,z_2)$. Fix a piecewise smooth path in N from z_1 to z_2. Let $Pt(c)$ denote the parallel transport along c and $\tau : M \times_N M \to S^1$ the transition map along the fibers $\tau(x_1,x_2) = s$ iff $x_2 = s.x_1$. The quantomorphisms commute with the parallel transport: $\varphi \circ Pt(c) = Pt(\psi\circ c)\circ\varphi$. Using this result we get:

$$\Psi_2 \circ \Psi_1^{-1}(\psi,s) = (\psi, \tau(Pt(c)(x_1),x_2)\tau(h_2(\psi(z_2)), Pt(\psi\circ c)(h_1(\psi(z_1))))s),$$

hence a smooth dependence on ψ and s. The principal bundle structure given by the atlas $\{\Psi\}$ induces canonically a smooth manifold structure which makes $\mathrm{Diff}(M,\alpha)_0$ a convenient Lie group. Indeed, the group operations are smooth. This is shown by using again the commutativity of quantomorphisms with the parallel transport and the smooth dependence of the parallel transport on the curve

$$Pt : C^\infty(\mathbb{R}, N) \times_{(ev_0,\pi)} M \to M.$$

Finally, $\mathrm{Diff}(M,\alpha)_0$ is regular because it is a group of diffeomorphisms. $\square$

All the groups involved in (1') and (2') are regular convenient Lie groups. The same is true for the exact sequence of universal covering groups of (2'):

$$0 \longrightarrow \mathbb{R} \xrightarrow{\tilde{j}} \widetilde{\mathrm{Diff}}(M,\alpha)_0 \xrightarrow{\tilde{p}} \widetilde{\mathrm{Ham}}(N,\omega) \longrightarrow 0. \tag{3}$$

Theorem. [KM] *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a bounded Lie algebra homomorphism. If H is regular and if G is simply connected then there exists a unique homomorphism $f : G \to H$ of Lie groups with $T_e f = F$.*

There exists a unique homomorphism of Lie groups $\tilde{s} : \widetilde{\mathrm{Ham}}(N,\omega) \to \widetilde{\mathrm{Diff}}(M,\alpha)_0$ such that $T_{id}\tilde{s} = S$. The explicit construction of $\tilde{s}$ is: $\tilde{s}(\{\psi_t\}) = \{\varphi_t\}$ where the isotopy φ_t is uniquely determined by $\varphi_0 = id$ and $\dot{\varphi}_t = S(\dot{\psi}_t)$. Applying the unicity part in the theorem for $PS = id_{\mathcal{X}_{Ham}}$ we get $\tilde{p}\tilde{s} = id_{\widetilde{Ham}(N,\omega)}$, so $\tilde{s}$ is a splitting section for (3).

Proposition 5. *The universal covering of the identity component of the quantomorphism group is a direct Lie group product of the universal covering of the group of Hamiltonian diffeomorphisms on N and $\mathbb{R}$*

$$\widetilde{\mathrm{Diff}}(M,\alpha)_0 = \widetilde{\mathrm{Ham}}(N,\omega) \times \mathbb{R}.$$

Another restriction of the exact sequence (1') is

$$0 \longrightarrow C^\infty(N, S^1) \xrightarrow{j} \mathrm{Diff}_{S^1}(M, d\alpha) \xrightarrow{p} \mathrm{Diff}(N,\omega) \longrightarrow 0$$

with the infinitesimal version

$$0 \longrightarrow C^\infty(N, \mathbb{R}) \xrightarrow{J} \mathcal{X}_{S^1}(M, d\alpha) \xrightarrow{P} \mathcal{X}(N,\omega) \longrightarrow 0,$$

where $\mathrm{Diff}_{S^1}(M, d\alpha)$ is the group of those S^1-equivariant diffeomorphisms which preserve the 2-form $d\alpha$ and $\mathcal{X}_{S^1}(M, d\alpha)$ is the Lie algebra of S^1-invariant vector fields X on M with $\mathcal{L}_X d\alpha = 0$. This exact sequence of Lie groups is important because there is an extension to $\mathrm{Diff}_{S^1}(M, d\alpha)$ of the flux homomorphism on $\mathrm{Diff}(N,\omega)$, see [B].

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