Parallel Immersions into Spaces of Constant Curvature and Conformal Transformations

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Abstract

The exterior parallelism of immersions in the sense of H.R. Farran and S.A. Robertson [J. London Math. Soc. (2) **35** (1987), 527–538] is studied in the case of ambient spaces of constant curvature. The different types of parallel ranks are generalized to this situation. Their local version is shown to be closely related with the curvature structure of certain subbundles of the normal bundle. It also has particular implications on the locus for the focal points of an immersion. Finally it is shown that the local parallel rank is preserved if the image of the immersion is subject to a local conformal transformation into possibly another ambient space of constant curvature. [1]

1 Preliminaries

The notion of *exterior parallelism* of two smooth immersions f_1, f_2 of the manifold M into Euclidean n-space E^n has been introduced by H.R. Farran and S.A. Robertson in [1] in the following sense: For every point $p \in M$ the affine spaces normal to $f_1(M)$ at $f_1(p)$ and $f_2(M)$ at $f_2(p)$ coincide. Having in mind that the axiom of planes characterizes spaces of constant curvature, it is easy to see that this notion can be transfered in an obvious way to ambient spaces of this type, and that it does not make sense to consider it for more general ambient Riemannian spaces, because they do not possess sufficiently many totally geodesic submanifolds.

Let $f : M \longrightarrow C$ be a smooth immersion of the n-dimensional manifold M into the m-dimensional space C of constant curvature, let $f_* : TM \longrightarrow TC$ denote the map induced in the corresponding tangent bundles. Let ∇ be the Levi-Civita connection of C, Nf the normal bundle of f, and $\nu_p f := E(p, N_p f)$ the analogue of the normal affine space in this more general case, obtained by applying the endpoint map

$$E : Nf \longrightarrow C, \quad E(p,v) := exp_{f(p)}(v),$$

(exp denoting the exponential map of C) to the normal vector space fibre of Nf at p; $\nu_p f$ is a totally geodesic k-dimensional submanifold of C where $k = m - n$ is the codimension of f. We may call it the normal subplane of f at p or $f(p)$.

Take a tangent vector field X to M and a normal vector field ξ of f. The splitting

$$\nabla_{f_*(X)}\xi =: \nabla^N_X \xi + f_*(-A_\xi(X))$$

into normal parts $\nabla^N_X\xi$ and tangential parts $f_*(-A_\xi(X)$ defines the normal connection ∇^N of f and the $(1,1)$-tensor field A_ξ of shape operators of f with respect to ξ. Taking a

[1] This paper is in final form and no version will be submitted elsewhere

second tangent vector field Y of M we get the Levi-Civita connection D of the induced metric on M with the same type of splitting

$$\nabla_{f_*(X)} f_*(Y) = f_*(D_X Y) + \sum_{i=1}^{k} < II(X,Y), e_i > e_i$$

where $< \cdot, \cdot >$ denotes the Riemannian metric of C and $e_1, \ldots, e_k$ is an orthonormal frame field in Nf. The normal vector valued second fundamental tensor II of f is related with the shape operators by

$$< II(X,Y), e_i > = < A_{e_i}(X), Y >,$$

where we do not distinguish between the notations of the metric on C and its induced metric on M. Finally the curvature of the normal bundle of f is characterized by its curvature transformations

$$R_p^N(X_p, Y_p) : N_p f \longrightarrow N_p f,$$
$$R_p^N(X_p, Y_p)\xi_p = \nabla^N_{X_p} \nabla^N_Y \xi - \nabla^N_{Y_p} \nabla^N_X \xi - \nabla^N_{[X,Y]_p} \xi,$$

$[\cdot, \cdot]$ denoting the Lie bracket.

2 Parallel immersions and parallel ranks

Definition 1 *a) Two immersions $f, g : M \longrightarrow C$ are called parallel($g \parallel f$), if for all $x \in M$ $\nu_x f = \nu_x g$.*
b) A self-parallelism of the immersion $f : M \longrightarrow C$ is given by a smooth diffeomorphism $\delta : M \longrightarrow M$ such that f and $f \circ \delta$ are parallel.

Using well-known computations for immersions into spaces of constant curvature, as they had been developed by B. Rämer [3] for example, we see the validity of the following criterion in the same way as its Euclidean version had been proved in [4].

Theorem 1 *The two immersions $f, g : M \longrightarrow C$ are parallel, if and only if there exists a parallel section ξ of the normal bundle of f such that $g(x) = E(x, \xi(x))$ $(= exp_{f(x)}(\xi(x)))$ for all $x \in M$.*

As immediate consequence we get the subsequent corollary.

Corollary 1 *a) If $f, g : M \longrightarrow C$ are parallel and $f(x) = g(x)$ for one $x \in M$, then $f = g$.*
b) For given f the evaluation of maps at any $x \in M$ is an injective mapping from $\{g \mid g \parallel f\}$ into $\nu_x f$.

Let $P(f)$ be the smallest totally geodesic subspace of $\nu_x f$ containing the image $\{g(x) \mid g \parallel f\}$ into $\nu_x f$ of this mapping and let $\rho(f)$ be the dimension of $P(f)$. According to Theorem 1 this notion does not depend on x. Denote by $\mu(f)$ the dimension of the subbundle of Nf generated by its parallel sections. Then the uniqueness of parallel transfer implies with Theorem 1

$$\rho(f) \leq \mu(f) \leq k.$$

The first inequality becomes an equality, if M is compact. There are simple non-compact examples where this inequality is strict.

Let us call $\rho(f)$ the *parallel rank* of f. Like in [1] and [4] we may introduce the *universal parallel rank* $\tilde{\rho}(f) := \rho(f \circ \eta)$, η denoting the universal covering of M, and the *uniform local parallel rank* $\rho_{loc}(f) := \min\{\rho(f, x) \mid x \in M\}$ where $\rho(f, x) := \max\{\rho(f|U) \mid x \in U \text{ open}\}$ is the *local parallel rank* of f at x.

Remark 1: Obviously we have the inequalities $\rho(f) \leq \tilde{\rho}(f) \leq \rho_{loc}(f) \leq \rho(f, x)$ where examples for the strict inequalities already have been given in the case of ambient Euclidean spaces [4]. Furthermore we notice that $\rho(f, x) = \rho(f|U) = \mu(f|U)$ for some suitable small neighborhood around x in M ($x \in M$ arbitrary).

An immersion $f : M \longrightarrow S^{m-1}$ into the round $(m-1)$-sphere simultaneously may be considered as an immersion $inc(S^{m-1}, E^m) \circ f =: \hat{f}$ into Euclidean m-space. Since a radial blowing up of S^{m-1} leads to a parallel mate for $\hat{f}$ which is not located in S^{m-1}, we easily conclude

Theorem 2 *For an immersion $f : M \longrightarrow S^{m-1}$ we have $\rho(f)+1 = \rho(inc(S^{m-1}, E^m) \circ f)$. The same is valid for the other types of parallel ranks.*

3 Local parallel ranks and the curvature of the normal bundles

Since the local parallel rank at x of the immersion $f : M \longrightarrow C$ is given by $\mu(f|U)$ for some neighborhood U of x, we may use the methods developed in the work of K. Yano and S. Ishihara [7] and [5] to relate this number with the structure of the normal bundle of f. The first of these results reads as follows:

Theorem 3 $\rho(f, x) \geq l \Leftrightarrow$ *There exists an open neighborhood U of x in M and an l-dimensional subbundle $\hat{N}$ of $Nf|U$ satisfying*
a) $\hat{N}$ is parallel with respect to the normal connection of f,
b) $R^N_y(X, Y) = 0$ for all $y \in U$, $X, Y \in T_y M$.

Here the condition that $\hat{N}$ is parallel with respect to the normal connection means that normal parallel transfer of a vector $\zeta_0 \in \hat{N}_y$ along a curve in M through y leads to a section of $\hat{N}$ along this curve.

Proof. I: Assume $\rho(f,x) \geq l$. Then there exists an open neighborhood U of x and (w.l.o.g.) an orthonormal frame field $\xi_1, \ldots, \xi_l$ in $Nf|U$ such that the ξ_i are parallel with respect to the normal connection (see Remark 1). Let $\hat{N}$ be the subbundle of Nf generated by $\xi_1, \ldots, \xi_l$. Then we have:
a) A parallel section ζ in the normal bundle $Nf|U$ along a curve α through $y \in U$ having initial value $\zeta_0 = \sum_{i=1}^{l} c_i \xi_i(y) \in \hat{N}_y$ necessarily has the form $\zeta(t) = \sum_{i=1}^{l} c_i \xi_i(\alpha(t))$ with real constants c_i. Hence this section remains in $\hat{N}$.
b) Using the same argument like in a) any $\zeta_0 \in \hat{N}_y$, $y \in U$, can be extended to a parallel section ζ over all of U, i.e. $\nabla^N_{\cdot}\zeta = 0$ which immediately implies $R^N_y(X,Y)\zeta_0 = 0$ for all $X, Y \in T_yM$.

II: Assume that conditions a) and b) above are satisfied. Consider some $\zeta_0 \in \hat{N}_x$ and local coordinates $u_1, \ldots, u_n$ around x, associated with a parametrization $\Phi : (-\varepsilon, \varepsilon)^n \longrightarrow U \subset M$, $\Phi(0, \ldots, 0) = x$. Let ζ_1 be the normal vector field obtained by parallel transfer with respect to the normal connection of ζ_0 along the first coordinate line through $x = \Phi(0, \ldots, 0)$. By assumption a) ζ_1 is a section of $\hat{N}$ along this curve. Having constructed ζ_h as a section of $\hat{N}$ along $\Phi((-\varepsilon,\varepsilon)^h \times \{(0, \ldots, 0)\})$ such that for all $i = 1, \ldots, h$ $\zeta_i = \zeta_h|\Phi((-\varepsilon,\varepsilon)^i \times \{(0, \ldots, 0)\})$ is parallel in the normal bundle along the lines of the i-th coordinates through $\Phi((-\varepsilon,\varepsilon)^{i-1} \times \{(0, \ldots, 0)\})$, we define ζ_{h+1} by extending ζ_h through normal parallel transfer along the lines of the $(h+1)$-th coordinate lines through $\Phi((-\varepsilon,\varepsilon)^h \times \{(0, \ldots, 0)\})$. By assumption a) ζ_{h+1} is a section of $\hat{N}$, and the properties of ζ_h extend to ζ_{h+1} accordingly. This finally leads to a section $\zeta := \zeta_n$ over $\Phi((-\varepsilon,\varepsilon)^n)$ which extends ζ_0 in a smooth way and which satisfies

$$\nabla^N_{\frac{\partial}{\partial u^i}} \zeta|_{\Phi(u_1, \ldots, u_i, 0, \ldots, 0)} = 0$$

for all $i = 1, \ldots, n$. The integrability condition under b) leads to the conclusion that ζ is parallel with respect to the normal connection as follows: On $\Phi((-\varepsilon,\varepsilon)^2 \times \{(0, \ldots, 0)\})$ we get

$$0 = R(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2})\zeta_2 = \nabla^N_{\frac{\partial}{\partial u^1}} \nabla^N_{\frac{\partial}{\partial u^2}} \zeta_2 - \nabla^N_{\frac{\partial}{\partial u^2}} \nabla^N_{\frac{\partial}{\partial u^1}} \zeta_2 = -\nabla^N_{\frac{\partial}{\partial u^2}} \nabla^N_{\frac{\partial}{\partial u^1}} \zeta_2 \ ,$$

which implies that $\nabla^N_{\frac{\partial}{\partial u^1}} \zeta_2$ is parallel along the u_2-lines in $\Phi((-\varepsilon,\varepsilon)^2 \times \{(0, \ldots, 0)\})$. Every such line meets $\Phi((-\varepsilon,\varepsilon) \times \{(0, \ldots, 0)\})$ where the restriction of ζ_2 is ζ_1. Hence $\nabla^N_{\frac{\partial}{\partial u^1}} \zeta_2$ vanishes there, which then implies that $\nabla^N_{\frac{\partial}{\partial u^1}} \zeta_2 = 0$ on all of $\Phi((-\varepsilon,\varepsilon)^2 \times \{(0, \ldots, 0)\})$. By construction we have $\nabla^N_{\frac{\partial}{\partial u^2}} \zeta_2 = 0$ there, hence $\nabla^N_{\cdot} \zeta_2 = 0$ on this set. This argument easily can be extended by induction to show that $\nabla^N_{\cdot} \zeta = 0$ on all of $\Phi((-\varepsilon,\varepsilon)^n)$.

Starting the same construction for an orthonormal basis of $\hat{N}_x$ we arrive at l orthonormal sections of $\hat{N}$ over a neighborhood of x which are parallel in the normal bundle of f. According to Remark 1 this implies $\rho(f,x) \geq l$.

Corollary 2 *$\rho(f,x) = k$ (k codimension of f) $\Leftrightarrow$ The normal curvature tensor of f vanishes on a neighborhood of x.*

Remark 2: It is known from the literature that the vanishing of the normal curvature tensor in some neighborhood of x implies the local existence of a parallel basic orthonormal frame field. This usually is proved in terms of differential forms. But the proof in terms of vector fields is not very hard as can be seen from the argument above which has been adjusted to our slightly more general situation.

Furthermore, we obviously may characterize $\rho(f, x)$ by the dimension of the maximal subbundle $\hat{N}$ of Nf satisfying the conditions from Theorem 3.

For the sake of simplicity we reduce our subsequent considerations in this section to the case where the ambient space is Euclidean. Then we may use exactly the same arguments like in [5] to show the impact of the local parallel rank on the structure of the focal set. To derive the same result for immersions into general spaces of constant curvature, the reader is referred to [3] or related investigations again.

Theorem 4 *Let $f : M \longrightarrow E^m$ be an immersion with $\rho(f) = l > 0$. Let $\hat{N}$ be the l-dimensional parallel subbundle of Nf according to Theorem 3. Then for any $x \in M$ the set of focal points of f with basis x decomposes in $E(\hat{N}_x) \subset \nu_x f$ into a system of at most n $(l-1)$-dimensional affine subspaces of $E(\hat{N}_x)$.*

Proof. The argument is the same as in [5]. The existence of an orthonormal parallel l-frame along f in $\hat{N}$ and hence the Codazzi equations for the shape operators imply that shape operators associated with normals in $\hat{N}_x$ commute under composition. Hence their eigenvectors and eigenvalues behave in a way which implies that focal points on different normals at x in $\hat{N}_x$ lie on common affine planes, if the eigenvalues have been sorted accordingly.

4 Parallel ranks and conformal transformations

The content of this section goes back to discussions with G. Romani where the idea originated that parallel sections of the normal bundle may remain almost parallel, if the ambient space is subject to a conformal transformation. The precise result is given in Proposition 1.

Hence let us consider an immersion $f : M \longrightarrow C_1$ into a space C_1 of constant curvature, a domain $G_1 \subset C_1$ with $f(M) \subset G_1$ and a conformal transformation $\Phi : G_1 \longrightarrow G_2$, $G_2 \subset C_2$ into possibly another space of constant curvature. Then the Riemannian metrics $< \cdot, \cdot >_1$ on G_1 and $< \cdot, \cdot >_2$ on G_2 are related by

$$\varphi(p) < X, Y >_1 |_p = < \Phi_*(X), \Phi_*(Y) >_2 |_{\Phi(p)}$$

for some suitable positive smooth function φ. Let ∇^i denote the Levi-Civita connection for $< \cdot, \cdot >_i$, $i = 1, 2$. For the sake of simplicity let us assume (w.l.o.g.) that $G_1 = G_2$ is equipped with two conformally related metrics $< \cdot, \cdot >_i$, $i = 1, 2$, reducing Φ to the identity. For tangent fields X, Y to the ambient space we have for the relation between the metrics and their connections the so-called Koszul formula (see B. O'Neill [2]):

$$\begin{aligned} 2 < \nabla^i_V W, X >_i &= V < W, X >_i + W < X, V >_i - X < V, W >_i \\ &- < V, [W, X] >_i + < W, [X, V] >_i + < X, [V, W] >_i . \end{aligned}$$

This leads to the following relation between ∇^1 and ∇^2 for conformally related metrics

$$\nabla_V^2 W = \nabla_V^1 W + \frac{1}{2\varphi}(V(\varphi)W + W(\varphi)V - grad_1(\varphi) < V, W >_1).$$

Proposition 1 *Let $f : M \longrightarrow G_1 \subset C_1$ be an immersion, $\Phi : G_1 \longrightarrow G_2 \subset C_2$ a conformal map, C_i, $i = 1, 2$, spaces of constant curvature and let ξ be a nonvanishing parallel section in the normal bundle of f. Then $\Phi_*(\xi)/\|\Phi_*(\xi)\|_2$ is a parallel section in the normal bundle of $\Phi \circ f$ in C_2.*

Proof. W.l.o.g. we may assume the simplification made above. Since both metrics are conformally related, the change of metrics does not affect the splitting into normal and tangential parts. Let ξ be a nonvanishing section of Nf which is parallel with respect to $\nabla^{1,N}$, and let prn denote the projection of tangent vectors to C_1 to normal parts. Then we get from the relation between ∇^1 and ∇^2 for any tangent vector V to M

$$\begin{aligned}
&\nabla_V^{2,N} \xi = prn(\nabla_V^2 \xi) = \\
&prn(\nabla_V^1 \xi + \frac{1}{2\varphi}(V(\varphi)\xi + \xi(\varphi)V - grad_1(\varphi) < V, \xi >_1)) = \\
&\frac{V(\varphi)}{2\varphi}\xi.
\end{aligned}$$

Hence $< \nabla_V^2(\xi/\|\xi\|_2), \xi/\|\xi\|_2 >_2 = 0$ implies

$$\begin{aligned}
0 \quad = \quad & < \nabla_V^{2,N} \frac{\xi}{\|\xi\|_2}, \frac{\xi}{\|\xi\|_2} >_2 = < (V(\frac{1}{\|\xi\|_2}) + \frac{V(\varphi)}{2\varphi\|\xi\|_2})\xi, \frac{\xi}{\|\xi\|_2} >_2 = \\
& (V(\frac{1}{\|\xi\|_2}) + \frac{V(\varphi)}{2\varphi\|\xi\|_2})\|\xi\|_2,
\end{aligned}$$

giving

$$V(\frac{1}{\|\xi\|_2}) + \frac{V(\varphi)}{2\varphi\|\xi\|_2} = 0,$$

hence

$$\nabla_V^{2,N} \frac{\xi}{\|\xi\|_2} = 0.$$

Theorem 5 *a) The local and uniform local parallel ranks of immersions into spaces of constant curvature are preserved under conformal transformations between spaces of constant curvature.*
b) The parallel rank of an immersion of a compact manifold into a space of constant curvature is preserved under conformal transformations between spaces of constant curvature.

Proof. This is a direct consequence from Proposition 1, according to Remark 1 for part a) and for part b) because for compact manifolds we have $\mu(f) = \rho(f)$.

Remark 3: There are examples of immersed open curves and conformal transformations of the Euclidean plane for which the conclusion in part b) of Theorem 5 is false.

In [6] it had been shown that for closed regular curves f in Euclidean 3-space $\rho(f) = 2$, if and only if the rotation of normal vectors under normal parallel transfer along one period of f (normal twist) is an integer multiple of 2π. This normal twist conicides with the total torsion for Frenet curves. Hence Theorem 5 immediately leads to

Corollary 3 *Let $f : S^1 \longrightarrow E^3$ be an immersed closed curve for which the normal twist (total torsion) is an integer multiple of 2π. Then the same is true for any image of the curve under a conformal transformation of the ambient space.*

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