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On a problem of A. Lupaş

Octavian Agratini

1 Introduction

At the International Dortmund Meeting held in Witten (Germany, March, 1995), A. Lupaş [4] formulated the following problem. " *Starting with the identity*

$$(1)\qquad \frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1,$$

let $\alpha = nx,\ x \geq 0$, *and consider the linear positive operators*

$$(L_n f)(x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right), \quad x \geq 0,$$

with $f : [0, \infty) \to \mathbf{R}$. *If we impose that* $L_n e_1 = e_1$ *we find* $a = 1/2$. *Therefore*

$$(2)\qquad (L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

This L_n-operator has a form very similar with Szász-Mirakyan operators. We have $L_n h = h,\ h \in \Pi_1$ and

$$\lim_{n\to\infty}(L_n e_2)(x) = e_2(x).$$

Find other properties of $L_n f$."

The focus of this note is to investigate these operators. An asymptotic formula and some quantitative estimates for the rate of convergence are given. By using a probabilistic method, this sequence is reobtained. Also two modified sequences are constructed.

2 The results

Firstly, we recall the common notation

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1), \quad k \geq 1.$$

For any real $x \geq 0$ and integer $r \geq 0$ we set

$$e_r(x) := x^r, \quad \psi_{x,r}(t) := (t-x)^r \quad (t \geq 0),$$

$$\mu_{n,r}(x) := (L_n \psi_{x,r})(x).$$

At this point, it has been proved that $L_n e_r = e_r, \quad r \in \{0,1\}$.

Remarks. 1) We can consider that $L_n,\ n \geq 1$, are defined on E where

$$E = \bigcup_{a>0} E_a$$

and E_a is the subspace of all real valued continuous functions f on $[0, \infty)$ such as

$$e(f; a) := \sup_{x \geq 0}(\exp(-ax)|f(x)|) < \infty.$$

The space E_a is endowed with the norm $\|f\|_a = e(f; a)$ with respect to which it becomes a Banach lattice.

2) In our investigations we also need to consider the Banach lattice $C_B[0, \infty)$ of all real-valued bounded continuous functions on $[0, \infty)$ endowed with the sup-norm $\| \cdot \|_\infty$. The operator L_n maps $C_B[0, \infty)$ into itself, it is continuous with respect to the sup-norm and $\|L_n\| = \|L_n e_0\|_\infty = 1$.

Lemma 1. *If L_n is defined by (2) then, for each $x \geq 0$, the following identities are valid*

$$(L_n e_2)(x) = x^2 + \frac{2x}{n}, \quad \mu_{n,2}(x) = \frac{2x}{n}.$$

Theorem 1. *If L_n is defined by (2) then one has*

$$\lim_{n\to\infty} L_n f = f \text{ uniformly on } [0,b],$$

for any $b > 0$.

Theorem 2. *Let L_n be defined by (2) and $b > 0$. One has*

$$|(L_n f)(x) - f(x)| \leq \left(1+\sqrt{2b}\right)\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \quad x \in [0,b].$$

If f has a continuous derivative on $[0,b]$ then

$$|(L_n f)(x) - f(x)| \leq \frac{2b+\sqrt{2b}}{\sqrt{n}}\omega_1\left(f';\frac{1}{\sqrt{n}}\right), \quad x \in [0,b].$$

In the following we are going to prove another estimate by involving the second order modulus of smoothness. In fact, our estimate will be based upon a more general theorem which is due to Gonska ([3], Theorem 4.1, page 331).

Theorem 3. *Let L_n be defined by (2) and $b > 0$. The following inequality*

$$|(L_n f)(x) - f(x)| \leq \left(3+2b\max\left(1,\frac{b}{n}\right)\right)\omega_2\left(f;\frac{1}{\sqrt{n}}\right),$$

$x \in [0,b]$, holds.

Also we establish a Voronovskaja-type formula.

Theorem 4. *Let $f \in C[0,\infty)$ be twice differentiable at some point $x > 0$ and let us assume that $f(t) = \mathcal{O}(t^2)$ as $t \to \infty$. If the operators L_n are defined by (2) then*

$$\lim_{n\to\infty} n((L_n f)(x) - f(x)) = \frac{\varphi^2(x)}{2} f'(x), \tag{3}$$

holds, where φ is defined by $\varphi(x) = \sqrt{2x}$, $x \geq 0$.

Actually, φ represents the step weight functions of the Lupaş operators and it controls their rate of convergence.

It is known that by using some concepts of the probability theory have been obtained several classical positive and linear operators. Pioneers in this field to be mentioned here are W. Feller [2] and D.D. Stancu [6].

Let $(X_{j,x})_{j\geq 1}$ be a sequence of independent random variables identically distributed

$$P(X_{j,x} = k) = 2^{-x-k} \frac{(x)_k}{k!}, \quad k \geq 0, \tag{4}$$

where x is a positive real parameter. Denoting by θ the common characteristic function of these random variables, the identity (1) implies

$$\theta(t) = \sum_{k=0}^{\infty} e^{itk} P(X_{j,x} = k) = (2 - e^{it})^{-x}.$$

If we set

$$Y_{n,x} := \frac{1}{n}\sum_{j=1}^{n} X_j \quad , \quad n \geq 1,$$

then the characteristic function of $Y_{n,x}$ will be $\phi_n(t) = \theta^n(t/n)$ which corresponds to the following distribution:

$$P(Y_{n,x} = k/n) = l_{n,k}(x)$$

where

$$l_{n,k}(x) := 2^{-nx}\frac{(nx)_k}{2^k k!}, \quad k \geq 0. \tag{5}$$

Furthermore, for every $n \geq 1$ and every $f \in E$ we consider the function $L_n f : [0,\infty) \to \mathbf{R}$ defined by $(L_n f)(x) := M(f \circ Y_{n,x})$ where $M(Z)$ represents the mathematical expectation of Z. This way we obtain the Lupaş operators.

As a matter of fact, all those approximation processes $(P_n)_{n\geq 1}$ of probabilistic type which are associated with a random scheme

$$Z_{n,x} = \frac{1}{n}\sum_{k=1}^{n} X_{k,x} \quad (n \geq 1,\ x \in I,\ X_{k,x} \text{ i.i.d.})$$

satisfy the formula

$$\lim_{n\to\infty} n((P_n f)(x) - f(x)) = \frac{\sigma^2(x)}{2} f'(x)$$

for every $f \in C_B^2(I)$, see [1, page 368]. Here $\sigma^2(x) = \mathrm{Var}(X_{k,x})$ represents the variance of $X_{k,x}$. For the variables $X_{n,k}$ defined by (4), we obtain

$$M(X_{k,x}) = x,\ M(X_{k,x}^2) = x^2 + 2x,\ \mathrm{Var}(X_{k,x}) = 2x.$$

So, in the particular case when $f \in C_B^2[0,\infty)$ we come across (3).

Theorem 5. *If $n \ge 1,\ k \ge 0,\ x \in (0,\infty)$ and $l_{n,k}(x)$ is defined by (5) then the following relations hold true*

i) $l_{n,k+1}(x) = \dfrac{nx+k}{2(k+1)} l_{n,k}(x)$.

ii) $l'_{n,k}(x) = n l_{n,k}(x) \left(\sum\limits_{i=0}^{k-1} (nx+i)^{-1} - \log 2 \right),\ (k \neq 0)$,

iii) $\int\limits_0^\infty l_{n,k}(x) = (n2^k k!)^{-1} \sum\limits_{i=0}^{k} (-1)^{k-i} s_{k,i} i! (\log 2)^{-i-1}$,

iv) $l_{n,k}(x) < \dfrac{4(2x^2+3x+2)}{\sqrt{nx}}$.

Here $s_{k,i}$ represents the Stirling numbers of the first kind.

3 Extensions

In order to obtain an approximation process in spaces of integrable functions, we introduce two integral modifications of these operators, Kantorovich-type operators

$$(K_n f)(x) = n \sum_{k=0}^{\infty} l_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt,$$

respectively Durrmeyer-type operators

$$(D_n f)(x) = \sum_{k=0}^{\infty} c_{n,k} l_{n,k}(x) \int_0^{\infty} l_{n,k}(u) f(u) du.$$

The coefficients $c_{n,k}$ are defined as follows

$$c_{n,k}^{-1} = \int_0^{\infty} l_{n,k}(x) dx.$$

In fact, this guarantees the relation $D_n e_0 = e_0$. Also, we easily obtain

$$(K_n e_0)(x) = 1, \quad (K_n e_1)(x) = x + \frac{1}{n},$$
$$(K_n e_2)(x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}.$$

As regards these integral operators we raise the problem to investigate their convergence in L_p-spaces.

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