General Mathematics
Vol. 6 /1998

R O G E R
1.06–3.06.1998, Sibiu-Romania

A conjecture related to the approximation operators of binomial type

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1 Terminology

A sequence $(p_n)_{n\geq 0}$ of polynomials is called a *polynomial sequence* iff degree$[p_n] = n$, $n = 0, 1, \ldots.$

Examples of such sequences , are $(p_0(x) := 1)$:

1. $e_n(x) = x^n$, (the monomials)

2. $a_n(x) = x(x - na)^{n-1}$, (Abel)

3. $(x)_n = x(x + 1)\ldots(x - n + 1)$, (upper-factorials)

4. $< x >_n = x(x - 1)\ldots(x - n + 1)$, (lower-factorials)

5. $t_n(x) = e^{-x}\sum\limits_{k=0}^{\infty}\dfrac{x^k k^n}{k!} = \sum\limits_{k=0}^{n}[0, 1, 2, \ldots, k; e_n]x^k$
   (Touchard or exponential polynomials)

6. $G_n(x) = G_n(x; a, b) = \frac{x}{x-an}\left\langle\frac{x-an}{b}\right\rangle_n$ (Gould)

All these sequences are of binomial type. A polynomial sequences $(p_n)_{n\geq 0}$ is called *binomial* , iff for all x, y

$$\boxed{b_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}b_k(x)b_{n-k}(y)} \quad , \quad n=0,1,2,\dots .$$

A polynomial sequence $\pi=(p_n)_{n\geq 0}$ which is binomial , has a generating function of the form

$$\boxed{e^{xh(t)}=\sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}} \quad , \tag{1}$$

where $h(t)$ is a formal power series

$$h(t)=\sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!} \quad ,$$

with $\beta_0=0$, $\beta_1\neq 0$.

At the same time, if $\pi=(p_n)_{n\geq 0}$ is generated by (1), then $\pi=(p_n)_{n\geq 0}$ is binomial.

Further, let us denote by Π the (real) linear space of all polynomials with real coefficients. Let us put in evidence some operators $\Pi\to\Pi$. For instance, I is the identity, D – the derivative , E^a is *the shift-operator* $(E^af)(x)=f(x+a)$. We shall use the notation

$$\Sigma=\{T:\Pi\to\Pi \quad , \quad T \text{ linear}\} \ ,$$

$$\Sigma_t=\{T\in\Sigma \ ; \quad E^aT=TE^a \quad , \ \forall a\} \ .$$

Definition 1 *An operator* Q *is called a* **delta operator**, *iff :*

$$Q \in \Sigma_t \text{ and } \quad Qe_1 = \text{const.} \neq 0 \ .$$

Further, let Σ_δ denotes `the set of all delta operators`. Examples of delta operators :

1) $Q = D(I - D)$

2) $\Delta_a = E^a - I = e^{aD} - I \ , \ \ a \neq 0 \quad ,$

3) $A = DE^a$ – Abel operator ,

4) $L = \frac{D}{D-I}$, - Laguerre operator

5) $G = E^{a+b} - E^a = \Delta_b E^a$ – Gould operator

6) $T = \ln(I + D)$ – Touchard operator .

Supposing $Q \in \Sigma_t$, then $Q \in \Sigma_\delta$ if and only if $Q = DP$, where $P \in \Sigma_t$, $Pe_0 \neq 0$. Likewise, $Q \in \Sigma_\delta$ iff $Q = f(D)$, $f(t)$ being a formal power series with $f(0) = 0$, $f'(0) \neq 0$.

Let X, $X \in \Sigma$, be defined as $(Xh)(t) = th(t)$, $h \in \Pi$. If T is a *shift-invariant operator*, i.e. $T \in \Sigma_t$, then the linear operator T' defined by $T' = TX - XT$, is the so-called

Pincherle derivative of T. It is known that

$$T \in \Sigma_t \Rightarrow T' \in \Sigma_t \quad , \qquad Q \in \Sigma_\delta \Rightarrow Q'^{-1} \text{ exists .}$$

Definition 2 *Let $Q \in \Sigma_\delta$. If*

$$b_0(x) = 1 \ , \ b_n(0) = 0 \ , \quad Qb_n = nb_{n-1} \ , \ n \geq 1 \ ,$$

then (b_n) is a **basic sequence associated** *to Q.*

Theorem 1 ([8] - [10])

i) Every delta operator has a unique sequence of basic polynomials.

ii) If (b_n) is a basic sequence for some delta operator Q, then it is binomial.

iii) If (b_n) is a binomial sequence, then it is a basic sequence for some delta operator.

iv) Let Q be a delta operator with basic sequence (b_n), and $Q = f(D)$. Let $f^{-1}(t)$ be the inverse formal power series. Then

$$\sum_{k=0}^{\infty} \frac{b_k(x)}{k!} t^k = e^{xh(t)} \quad , \quad h(t) = f^{-1}(t) \ ,$$

and $b_n(x) = (XQ'^{-1}b_{n-1})(x)$.

2 The class $\mathcal{B}$ of operators

In the following let us construct- (by means of the "umbral calculus")- sequences of approximation operators

$$L_n{}^Q : C(J) \to C(J)\ ,\ n = 0, 1, \ldots,\ J = [0, 1],$$

of the following form

$$(2)\quad \boxed{(L_n{}^Q f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n - nx) f\left(\frac{k}{n}\right)}$$

where $\pi = (p_n)_{n\geq 0}$ is the **basic sequence associated to** Q , $Q \in \Sigma_\delta$, $p_n(n) \neq 0$.

$$\boxed{\mathcal{B} = \{L_n{}^Q; Q \in \Sigma_\delta, L_n{}^Q \text{positive operators},\ n = 1, 2, ...}.$$

Theorem 2 (**T.Popoviciu [7]-1932**) *Let* $Q \in \Sigma_\delta,\ Q = f(D)$, *and* $L_n{}^Q$ *as in (2). Then*

$$(L_n{}^Q) \in \mathcal{B} \iff f^{-1}(t) = \sum_{k=1}^{\infty} c_k \frac{t^k}{k!},\ c_1 > 0,\ c_k \geq 0, k \geq 2.$$

see also: H.Brass [1] -1971, P.Sablonniere [11] – 1995.

Remark: The Bernstein operator B_n admits the representation (i.e., $(B_n) \in \mathcal{B}$)

$$(B_n f)(x) = \frac{1}{e_n(n)} \sum_{k=0}^{n} \binom{n}{k} e_k(nx) e_{n-k}(n - nx) f\left(\frac{k}{n}\right);$$

a). Some representations

Let us consider $Q \in \Sigma_\delta$, X, $Xf(t) = tf(t)$, $P = XQ'^{-1}$.

Theorem 3 *If* $T_{k,y} : \Pi \to \Pi$ *is defined as*

$$T_{k,y} = \frac{1}{k!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} P^{j+k} Q^{j+k} E^y \ , \quad k = 0, 1, ..., \ y \in \mathbf{R} \ .$$

and $\pi = (p_n)_{n \geq 0}$ *is the basic sequence of* Q*, then*

$$\binom{n}{k} p_k(x) p_{n-k}(y) = (T_{k,y} p_n)(x)$$

$$\sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) f(\frac{k}{n}) =$$

$$= \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} [0, \frac{1}{n}, ..., \frac{k}{n} \ ; \ f](P^k E^y p_{n-k})(x) \ .$$

Corollary 1 *If* ${L_n}^Q \in \mathcal{B}$ *and*

$$d_{k,n}(x) = \frac{1}{p_n(n)} (P^k E^{n-nx} p_{n-k})(nx) \ ,$$

then $d_{k,n} \in \Pi_k$, *degree*$(d_{k,n}) = k$*. Moreover*

$$({L_n}^Q f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} [0, \frac{1}{n}, ..., \frac{k}{n} \ ; \ f] d_{k,n}(x) \ .$$

Also ${L_n}^Q(\Pi_s) \subseteq \Pi_s$.

b). The convergence of $({L_n}^Q)_{n \geq 1}$

Further, $Q \in \Sigma_\delta$ is such that $({L_n}^Q) \in \mathcal{B}$. For $n \geq 2$, let

$$\boxed{W_n(Q) \;=\; 1 - \;\frac{n(n-1)}{p_n(n)}({Q'}^{-2}p_{n-2})(n)} \;\; .$$

Theorem 4 *The sequence* $(W_n(Q))_{n\geq 2}$ *is bounded. More precisely*

$$\frac{1}{n} \;\leq W_n(Q) \;\leq 1 \;\; .$$

Theorem 5 *Let* $({L_n}^Q) \in \mathcal{B}$. *Then*

$$\lim_{n\to\infty} \| \; f - {L_n}^Q f \; \| = 0, \; \forall f \in C(J), \;\; \Longleftrightarrow \;\; \lim_{n\to\infty} W_n(Q) = 0.$$

Moreover

$$|f(x) - ({L_n}^Q f)(x)| \;\leq 2.\omega(f; \sqrt{x(1-x)W_n(Q)}) \quad , \quad x \in J \; .$$

Further we shall put in evidence some favourable situations in which $({L_n}^Q)_{n\geq 1}$ are approximation operators.

Examples :

$Q = f(D)$	$p_n(x)$	$W_n(Q)$	$f^{-1}(t)$
D	$e_n(x)$	$W_n(D) = \frac{1}{n}$	t
$L = \frac{D}{I+D}$ (Laguerre)	$n!{L_n}^{-1}(-x)$	$\frac{1}{n} \leq W_n(L) \leq \frac{3}{n}$	$\frac{t}{1-t}$
$T = \ln(I+D)$ (Touchard)	$t_n(x)$	$\frac{1}{n} \leq W_n(T) \leq \frac{2}{n}$	$e^t - 1$
$\nabla = I - E^{-1}$ Backward difference	$(x)_n =$ $x...(x+n-1)$	$W_n(\nabla) = \frac{2}{n+1}$	$-\ln(1-t)$

c). The behaviour on the convex functions

Further denote by $\mathbf{K_s}$ $-$ the set of all f , $f \in C(J)$, which are non-concave of s-th order on J.

Theorem 6 *Let* ${L_n}^Q \in \mathcal{B}$. *Then*

1. ${L_n}^Q(\mathbf{K_s}) \subseteq \mathbf{K_s}$,

2. $B_n f \leq {L_n}^Q f$, $f \in \mathbf{K_1}$, *on* J .

Corollary 2 *Let* $f \in \mathbf{K_1}$ *and* $B_n = {L_n}^D$ *be the Bernstein* *operator. Then*

$$\min_{{L_n}^Q \in \mathcal{B}} \| f - {L_n}^Q f \| = \| f - B_n f \| \ .$$

and

$$f(x) \le (1 - \{nx\}) f(\frac{[nx]}{n}) + \{nx\} f(\frac{1 + [nx]}{n}) \le (B_n f)(x) \le$$
$$\le ({L_n}^Q f)(x) \ , \ x \in [0,1) \ .$$

Conjecture (first given at IDoMAT-1998) *If* $({L_n}^Q) \in \mathcal{B}$, *then there exists a sequence of linear operators* ${S_n}^Q : C(J) \to C(J), n = 1, 2, ...,$ *with*

$$f \in \mathbf{K_s} \implies {S_n}^Q f \in \mathbf{K_s} \ , \ s = -1, 0, 1, ...,$$

such that

$$\boxed{{L_n}^Q = {S_n}^Q B_n \ , \ \ on \ \ C(J)} \ , \quad B_n = {L_n}^D \ ,$$

where B_n *is the Bernstein operator.*

The reasons of this statement are :

1. The extremal property from Corollary 2,
2. There are some particular cases in which such representations are valid. For instance

a). Let $Q = T = \ln(I + D)$ - Touchard.

$$S_n{}^T = \frac{e^{-n}}{t_n(n)} \sum_{k=0}^{\infty} \frac{k^n n^k}{k!} B_k \quad .$$

b). $Q = \nabla$.

$$(S_n{}^\nabla f)(x) = \int_0^1 \frac{t^{nx-1}(1-t)^{n-nx-1}}{B(nx, n-nx)} f(t)\, dt \quad , \quad x \in (0,1) \ ,$$

$$(S_n{}^\nabla f)(x_0) \ := f(x_0) \quad , \quad x_0 \in \{0,1\} \ .$$

All the proofs are given in [4]

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