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The ϵ – variants of a prescribed sequence of orthogonal polynomials

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Let $d\alpha(x)$ be a positive measure supported and of bounded variation on the real line with finitely or infinitely points of increase such that the moments of $d\alpha$

$$\mu_k \ := \int_{-\infty}^{\infty} t^k \, d\alpha(x) \quad , \quad n = 1, 2, \dots \, ,$$

exist. Suppose that $[a, b]$ is the smallest closed interval that contains the support $Br(d\alpha)$ (i.e. the set of the points of increase) of α .

A polynomial sequence $(P_n)_{n\geq 0}$ is called *orthogonal with respect to* $d\alpha$ if and only if

$$\int_{-\infty}^{\infty} P_n(x)P_j(x) \, d\alpha(x) = h_n\delta_{n,j} \quad , \quad h_n \neq 0 \ \ .$$

One of the characteristic feature of orthogonal polynomials is the fact that every polynomial sequence $P = (P_n)_{n\geq 0}$ which is orthogonal satisfies a three term recurrence relation which we may write in the form

(1) $$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x) \quad ,$$

$$(\ P_0(x) = 1 \ , \quad P_{-1}(x) \ := \ 0 \ , n = 0, 1, 2, ...,) \ ,$$

with

(2) $$\boxed{A_n A_{n-1} C_n \ > 0} \ .$$

Further suppose that

$$P_n(x_{n,i}) \ = \ 0 \quad , \quad i = 1, 2, ..., n \ .$$

It is well-known that $x_{n,i}$ are simple roots and $x_{n,i} \in (a, b)$.

There is a converse result, which says that any sequence generated by a recurrence of the form (1) , with (2), is an orthogonal sequence.

This theorem is usually known as *Favard's theorem* – (see [1] -**1935**). More precisely, if $P = (P_n)_{n \geq 0}$ is a polynomial sequence which satisfies (1)-(2), then there exists a positive measure μ on the real line such that P is an orthogonal sequence with orthogonality measure μ . This measure need not be unique but a sufficient condition for a unique measure is

$$(3) \qquad \sum_{k=0}^{\infty} A_{k+1} = \infty \quad , \quad (Carleman's\ condition).$$

Regarding Favard$'$s theorem, it has often been pointed out that a form of this result is implicit in T.J.Stieltjes paper [3]–**1894** as well as in H.Hamburger [2] – **1920**.

Definition 1. *Suppose that $P = (P_n)$ is a sequence of orthogonal polynomials which satisfy (1). Further, let $\epsilon := (\epsilon_1, \epsilon_2, \epsilon_3)$ be a fixed system of positive numbers and consider three sequences*

$$(a_n) \quad , \quad (b_n) \quad , \quad (c_n)$$

with the properties

a). $|A_n - a_n| < \epsilon_1 \quad |B_n - b_n| < \epsilon_2 \quad , \quad |C_n - c_n| < \epsilon_3$;

b). $a_n a_{n-1} c_n > 0$;

c). $\sum_{k=0}^{\infty} a_{k+1} = \infty$.

If $\pi = (\pi_n)$ *is a polynomial sequence generated by*

$$(4) \qquad \pi_{n+1}(x) = (a_n x + b_n)\pi_n(x) - c_n \pi_{n-1}(x) \quad ,$$

$$(n = 0, 1, \ldots , \ \pi_0(x) = 1 \quad , \quad \pi_{-1}(x) = 0 \ )$$

then π *is called an* "ϵ"–**variant** *of the sequence* P .

Let us remark that according to Favard's theorem the sequence $\pi = (\pi_n)_{n\geq 0}$ is orthogonal with a certain measure $d\alpha^*$ with the corresponding support $Br(d\alpha^*)$. Also suppose that the smallest closed interval which contains $Br(d\alpha^*)$ is $[a^*, b^*]$. The roots of π_n are denoted by $y_{n,i}$, $i = 1, 2, ..., n$.

Supposing that we intend to generate the sequence $P = (P_n)_{n\geq 0}$ by means of (1). In fact, a computer used (4) which enables us to assert that instead of P we find an "ϵ"–variant $\pi = (\pi_n)_{n\geq 0}$. In conclusion, we call attention to the following

Problem 1.

- *Which is the relation between* $d\alpha$ *and* $d\alpha^*$*?*
- *How we can compare the intervals* (a, b) *and* (a^*, b^*)*?*

Problem 2. *Find estimates for* $|x_{n,i} - y_{n,i}|$.

Problem 3. *Let us denote*

$$\Delta_n(x) = {P_n}^2(x) - P_{n-1}(x)P_{n+1}(x) \quad ,$$

$$\delta_n(x) = {\pi_n}^2(x) - \pi_{n-1}(x)\pi_{n+1}(x) \quad .$$

If we suppose that $J \ := (a,b) \cap (a^*, b^*) \neq \emptyset$ *investigate if for* x *in an arbitrary compact subset of* J

$$\lim_{n\to\infty} \frac{\Delta_n(x)}{\delta_n(x)}$$

exists.

References

[1] J.Favard, *Sur les polynômes de Tchebicheff*, C.R.Acad.Sci.Paris **200** (1935), 2052-2053.

[2] H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math.Anal., **81** (1920), 253-319; **82** (1921), 120-164; 168-187.

[3] T.J.Stieltjes, *Recherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, **8**(1984), 1-122; 9(1895), 1-47.