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A proof of Leibniz's formula for divided differences

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Abstract

In this note we provide a simple proof of Leibniz's formula for divided differences.

1 Preliminaries and Notations

Let $x_0, \ldots, x_n \in \mathbf{R}$ (or $\mathbf{C}$) be distinct points and let f, g be real (or complex) valued functions defined on $\{x_0, \ldots, x_n\}$.

Definition 1. *The polynomial P of degree at most n having the property*

$$P(x_i) = f(x_i), \qquad i = 0, \ldots, n,$$

is called the Lagrange Interpolating Polynomial *of the function f with respect to* $x_0, \ldots x_n$.

It is known that the Lagrange Polynomial is unique. We denote it by the symbol $L(x_0, \ldots, x_n; f)$.

Definition 2. [DL93, p. 120, Eq. (7.1)] *The* divided difference *of the function f with respect to the knots $x_0, \ldots, x_n$, is defined to be the coefficient at x^n in the Lagrange interpolating polynomial $L(x_0, \ldots, x_n; f)$, and is denoted by $[x_0, \ldots, x_n;\ f]$.*

The divided difference can also be defined through the formula [DL93, p. 121, Eq. (7.7)]

$$(1) \qquad [x_0, \ldots, x_n;\ f] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} .$$

This obviously yields

$$(2) \quad [x_0, \ldots, x_n;\ L(x_0, \ldots, x_n; f) \cdot g] = [x_0, \ldots, x_n;\ f \cdot g].$$

If $x_0, \ldots, x_i, y_0, \ldots, y_j$ are distinct points, by simple calculation, formula (1) implies

$$(3) \qquad [x_0, \ldots, x_i, y_0, \ldots, y_j; (t - x_0) \cdots (t - x_i) f]_t = \\ = [y_0, \ldots, y_j; f] .$$

Formula (3) can also be deduced by identifying the coefficients at x^n in the following immediate identity

$$(4)\quad L(x_0,\ldots,x_i,y_0,\ldots,y_j;\,(t-x_0)\cdots(t-x_i)\,f(t))(x) =$$

$$= (x-x_0)\cdots(x-x_i)\,L(y_0,\ldots,y_j;\,f)(x).$$

By using divided differences, the Lagrange Polynomial can be written in Newton′s Form [DL93, p. 120, Eq. (7.2)]

$$(5)\qquad L(x_0,\ldots,x_n;f)(x) =$$

$$= \sum_{k=0}^{n}(x-x_0)\cdots(x-x_{k-1})\,[x_0,\ldots,x_n;\,f].$$

The divided difference of the product $f\cdot g$ can be expressed by the Leibniz formula [DL93, p. 121, Eq. (7.10)]

$$(6)\quad [x_0,\ldots,x_n;\,f\cdot g] = \sum_{k=0}^{n}[x_0,\ldots,x_k;\,f]\cdot[x_k,\ldots,x_n;\,g].$$

Proofs of this formula can be found, e.g., in [Pop40], [DeB78, p. 5], [DL93, p. 121]. The Leibniz formula can also be proven through mathematical induction. We will give bellow a simple proof of formula (6).

2 The Proof

By using (2), (3) and (5), we obtain successively:

$$[x_0,\ldots,x_n;\ f\cdot g]=[x_0,\ldots,x_n;\ L(x_0,\ldots,x_n;\ f)\cdot g]=$$

$$=\left[x_0,\ldots,x_n;\ \sum_{k=0}^{n}(t-x_0)\cdots(t-x_{k-1})[x_0,\ldots,x_k;\ f]\cdot g(t)\right]_t=$$

$$=\sum_{k=0}^{n}[x_0,\ldots,x_k;\ f]\cdot[x_0,\ldots,x_n;\ (t-x_0)\cdots(t-x_{k-1})\,g(t)]_t=$$

$$=\sum_{k=0}^{n}[x_0,\ldots,x_k;\ f]\cdot[x_k,\ldots,x_n;\ g].$$

References

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