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A sequence considered by I.J. Schoenberg?

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Let us consider the sequence $(s_n)_{n\geq 1}$ with

$$s_n = \frac{1^n + 2^n + \cdots + (n-1)^n}{n^n}.$$

We have obtained some estimations related to the sequence s_n.

If anybody knows who has investigated this sequence, please inform us. It was I.J.Schoenberg?

Comment by A.Lupaş: We shall use the notation from the monograph *"Divergent Series"* by G.H.Hardy [Oxford, 1949, see also russian edition, Moscow, 1951, pag. 393–428.]

Let us denote

$$B_n = \frac{(2n)!}{2^{2n-1}\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \quad .$$

These (modified) Bernoulli numbers are generated by means of

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \sum_{k=1}^{\infty} (-1)^{k-1} B_k \frac{t^{2k}}{(2k)!} \quad , \quad |t| < 2\pi \quad .$$

The first of these numbers are

$$B_1 = \frac{1}{6} \quad , \quad B_2 = \frac{1}{30} \quad , \quad B_3 = \frac{1}{42} \quad , \quad B_4 = \frac{1}{30} \quad \dots .$$

Now, let us define

$$A_n = \frac{7}{12} - \frac{1}{n+1} - \frac{1}{12\, n^{n-1}} - \frac{1}{n^n(n+1)} + \frac{1}{2\, n^n} \quad ,$$

$$C_n(p) = \sum_{j=2}^{p} \frac{(-1)^{j-1}B_j}{2j} \binom{n}{2j-1} \left(\frac{1}{n^{2j-1}} - \frac{1}{n^n} \right) \quad , \quad p = 2, 3, \dots .$$

By means of *"Euler-Maclaurin summation formula"* (see for instance the above quoted Hardy's monograph) one observes that the following inequalities are verified

$$(1) \quad A_n + C_n(2q) \ \leq s_n \ \leq A_n + C_n(2p+1) \quad , \quad n \geq 5 \ ,$$

where

$$1 \leq p \leq \left[\frac{n-1}{4} \right] \quad , \quad 1 \leq q \leq \left[\frac{n+1}{4} \right] \quad .$$

At the same time, it may be shown that

$$s_n = A_n + C_n \left(\left[\frac{n}{2} \right] \right) \quad .$$

Also, for a fixed p , $p \geq 2$, one has

$$C_n(p) = \sum_{j=2}^{p} \frac{(-1)^{j-1}B_j}{(2j)!} + \frac{1}{4n} \sum_{j=2}^{p} \frac{(-1)^{j-1}B_j}{j(2j-3)!} +$$

$$+ \frac{1}{24n^2} \sum_{j=2}^{p} \frac{(-1)^{j-1}(12j^2 - 22j + 9)B_j}{j(2j-3)!} + \mathcal{O}\left(\frac{1}{n^3} \right) \quad n \to \infty \ .$$