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Some problems regarding Bernstein Polynomials

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1. Let s be a natural number and denote by $\mathbf{K}_s[0,1]$ the cone of all functions $f:[0,1]\to\mathbf{R}$ which are non-concave of order s on $[0,1]$, i.e. for any system $x_0,x_1,...,x_{s+1}$, of distinct points from $[0,1]$, we have

$$[x_0,x_1,...,x_{s+1}\ ;f]\geq 0 \quad ,$$

where $[x_0,x_1,...,x_{s+1}\ ;f]$ denotes the divided difference of f at the specified points. If $-f\in\mathbf{K}_s[0,1]$ we shall say that $f\in\overline{\mathbf{K}}_s[0,1]$.

Further denote by B_n the Bernstein operator

$$(B_nf)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right) \quad ,$$

$$b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k} \quad , \quad f\in C[0,1] \ .$$

According to D.D. Stancu [4] it is known that if

$$f \in \mathbf{K}_1[0,1] \cap \mathbf{K}_2[0,1],$$

then

$$(1) \qquad (B_n f)'(x) - (B_{n+1} f)'(x) \geq 0 \quad , \quad \forall x \in \left[0, \frac{1}{2}\right] .$$

Further, it was proved that (see [3])

$$[x, y\, ; B_n f - B_{n+1} f] =$$

$$(2) \qquad \begin{aligned} &= \frac{1-x-y}{n(n+1)} \sum_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}\, ; f\right] b_{n-1,k}(y) + \\ &+ \frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-2} \left\{ \frac{2}{n} \left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n+1}, \frac{k+2}{n}\, ; f\right] + \right. \\ &\left. + \frac{1}{n+1} \left[\frac{k+1}{n+1}, \frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n+1}\, ; f\right] \right\} [x, y\, ; Q_{k,n-1,1}] \end{aligned}$$

where

$$Q_{k,n,0}(x) = b_{n,p}(x)\ ,\ Q_{k,n,p}(x) = \sum_{i=k+p}^{n} \binom{i-k-1}{p-1} b_{n,i}(x)$$

$$p \geq 1.$$

Since for $p \in \{0, 1, ..., n\}$

$$(3) \quad D^p Q_{k,n,p} = \frac{p!}{(n-p)!} b_{n-p,k} \quad , \quad k = 0, 1, ..., n-p ,$$

D being the derivative, one observes that $Q_{k,n,p} \in \mathbf{K}_{p-1}[0,1]$, $p \geq 1$. Taking into account (1)-(3) it follows the following extension of D.D. Stancu result [3] : *If*

$$f \in \mathbf{K}_1[0,1] \cap \mathbf{K}_2[0,1],$$

then for all

$$(x,y) \in \Delta := \{(x,y) \in [0,1] \times [0,1]; x+y \leq 1\}$$

the inequalities

$$[x,y;B_n f] \geq [x,y;B_{n+1} f] \quad , \quad n = 1,2,\ldots .$$

are satisfied.

It is natural to ask whether the converse statement is true.

Problem 1 $f \in \mathbf{K}_1[0,1] \cap \mathbf{K}_2[0,1]$ *if and only if for all* $(x,y) \in \Delta$

$$[x,y\,;B_n f)] \geq [x,y\,;B_{n+1} f] \quad , \quad n = 1,2,\ldots .$$

2. Another problem which may be proposed in connection with the theory of approximation by means of Bernstein operators is the following.

It is well known that

$$f \in \mathbf{K}_1[0,1] \implies f(x) \leq (B_n f)(x) \quad , \quad \forall\, x \in [0,1]\,.$$

Problem 2 *Let* $f \in \mathbf{K}_s[0,1]$, $s \geq 2$. *Whithout using derivatives, what inequalities satisfies the Bernstein polynomial* $B_n f$*?*

Let us consider an example, namely $s = 3$. Further we shall investigate if an inequality of the form

$$(4) \qquad c_1(x) f(x_1(x)) + c_2(x) f(x_2(x)) \leq (B_n f)(x) \quad ,$$

$$(\forall f \in \mathbf{K}_3[0,1]\,,\ x_1(x), x_2(x) \in [0,1]\,, x \in [0,1])$$

exists. By using the theory of convex functions founded by Tiberiu Popoviciu, the inequality (4) holds if and only if $(e_j(t) = t^j)$ for $j = 0,1,2,3,$

$$(5) \qquad c_1(x) e_j(x_1(x)) + c_2(x) e_j(x_2(x)) = (B_n e_j)(x) \quad ,$$

as well as

$$(6) \qquad c_1(x) h_x^\star(x_1(x)) + c_2(x) h_x^\star(x_2(x)) \leq (B_n h_x^\star)(x) \quad ,$$

$$\forall\, x \in [0,1]$$

where $h_x^\star(t) \ := \ |t-x|^3_{\ +}$.

In order to prove the inequality (6) we meet many difficulties.

Firstly let us solve (5).

Lemma 1 *Suppose that x is fixed in $[0,1]$. There exists a unique system of numbers (x_1, x_2, c_1, c_2), $x_j \in [0,1]$, $j = 1, 2$, such that*

$$c_1 e_j(x_1) + c_2 e_j(x_2) = (B_n e_j)(x) \quad , \quad \textit{for } j = 0, 1, 2, 3 \; .$$

Moreover, if we denote

$$\varphi_n(x) = \sqrt{\frac{x(1-x)}{n} + \frac{(1-2x)^2}{4n^2}} \;\; ,$$

then the solution is

$$\begin{aligned} x_1 = x_1(x) &= x + \frac{1-2x}{2n} - \varphi_n(x) \; , \\ x_2 = x_2(x) &= x + \frac{1-2x}{2n} + \varphi_n(x) \; , \\ c_1 = c_1(x) &= \frac{1}{2} + \frac{1-2x}{4n\varphi_n(x)} \; , \\ c_2 = c_2(x) &= \frac{1}{2} - \frac{1-2x}{4n\varphi_n(x)} \; . \end{aligned}$$

It is interesting to remark the following inequalities:

$$0 \le x_1(x) \le x \le x_2(x) \le 1 \quad , \quad c_j(x) \ge 0 \; x \in [0,1] \; .$$

Likewise, if we define on $[0,1]$ the non-negative functions

$$\delta_n(x) = \sqrt{\frac{x(1-x)}{n}} \quad , \quad \Delta_n(x) = \frac{1}{n} + \sqrt{\frac{x(1-x)}{n}} \;\; ,$$

on the interval $[0,1]$ it may be shown that

$$\delta_n(x) \le \varphi_n(x) \le \Delta_n(x) \le \frac{8}{3}\varphi_n(x),$$

$$\varphi_n(x) \le \delta_n(x) + \frac{|1-2x|}{2n} \quad ,$$

$$\max_{x\in[0,1]} \delta_n(x) = \max_{x\in[0,1]} \varphi_n(x) = \varphi_n\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{n}}.$$

Further , $x_j, c_j, j = 1,2,$ are as above and consider the remainder $r_n(f,x)$ in the "discrete"-quadrature formula

$$(B_n f)(x) = c_1(x) f(x_1(x)) + c_2(x) f(x_2(x)) + r_n(f,x),$$

$$f \in C[0,1].$$

Consider $f \in C^4[0,1]$ and denote by h, $h \in \Pi_3$ the Hermite interpolation polynomial which satisfies the conditions

$$h(x_i) = f(x_i) \quad , \quad h'(x_i) = f'(x_i), \; i = 1,2.$$

Since there exists $\Theta = \Theta(t,n,x,f) \in [0,1]$ such that

$$f(t) = h(t) + W(t)\frac{f^{(4)}(\Theta)}{4!} \quad ,$$

$$W(t) = (t-x_1)^2(t-x_2)^2,$$

we find

$$(B_n f)(x) = (B_n h)(x) + B_n(W(t)\frac{f^{(4)}(\Theta)}{4!}; x).$$

Taking into account that

$$r_n(h,x)=0 \quad , \quad (B_nW)(x)=\frac{2(n-1)}{n^3}x^2(1-x)^2,$$

we have proved the following mean-value theorem.

Theorem 1 *Suppose that* x_1, x_2, c_1, c_2 *are defined as in Lemma . For each* $f \in C^4[0,1]$ *there exists in* $[0,1]$ *at least a point* $\zeta = \zeta(n,x)$ *such that*

$$\begin{aligned}(B_nf)(x) &= c_1(x)f(x_1(x)) + c_2(x)f(x_2(x))+ \\ &+\frac{n-1}{12n^3}x^2(1-x)^2 f^{(4)}(\zeta).\end{aligned}$$

In conclusion, in the case $s=3$, a solution of the Problem is the following

Theorem 2 *If* $f \in \mathbf{K}_3[0,1]$, *then for* $x \in [0,1]$

$$c_1(x)f(x_1(x)) + c_2(x)f(x_2(x)) \le (B_nf)(x) \quad , \quad n=1,2,\dots .$$

Proof. It is necessary to remark that Theorem asserts that the linear functional $A : C[0,1] \to \mathbf{R}$, $A(f) = r_n(f,x)$, n, x

fixed, has a so-called "simple form on the subspace $C^4[0,1]$ ". But according to [2] the functional has the simple form on the whole space $C[0,1]$: this means that for each continuous function f there exist a system $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ of distinct points from $[0,1]$ such that

$$A(f) = A(e_4)[\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\ f].$$

Since for each $x \in [0,1]$ the function $h_x^\star$ is in $\mathbf{K}_3[0,1]$ and

$$A(e_4) = r_n(e_4, x) = \frac{2(n-1)}{n^3} x^2(1-x)^2 \geq 0\ ,$$

it follows that (6) is valid. This completes the proof.

J.d-Cao [1] has proved that there exists a positive constant c such that for all $n \in \mathbf{N}, x \in [0,1], f \in C[0,1]$

$$|\,(B_n f)\,(x) - f(x)| \leq c\,\omega_2(f\ ; \delta_n(x))\ .$$

This result is very interesting, but it is hard to find the best constant c .

According to the relations between $\delta_n(x)$ and $\varphi_n(x)$ let us formulate an "intermediate" inequality.

Corollary 1 *If* $f \in \mathbf{K}_1[0,1] \cap \overline{\mathbf{K}}_3[0,1]$ *or* $f \in \overline{\mathbf{K}}_1[0,1] \cap$

$\mathbf{K}_3[0,1]$, *then*

$$|\,(B_n f)\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right)| \le \omega_2\left(f;\frac{1}{2\sqrt{n}}\right).$$

3. Let $(p_n)_{n\geq 0}$ be a polynomials sequence which is orthogonal on a certain interval, with respect to a certain weight. It is known that $B_n(\Pi_n) = \Pi_n$, Π_n being the linear space of all (real) polynomials of the degree at most n. This implies that there exists a polynomial f_n , such that

$$p_n(x) = (B_n f_n)(x) \quad , \quad n = 0, 1, ..., \; (B_0 f)(x) := f(0).$$

It is not difficult to give this polynomial, namely

$$f_n(t) = \sum_{k=0}^{n} (-1)^k \frac{(-nt)_k}{k!\,(-n)_k} p_n^{(k)}(0),$$
$$(a)_0 = 1 \quad , \quad (a)_k = a(a+1)...(a+k-1).$$

In other words, each sequence of orthogonal polynomials may be written as $(B_n f_n)_{n\geq 0}$.

Problem 3 *Find all bounded functions* $f : [0,1] \to \mathbf{R}$, $f(0) = 1$, *(if they exist) for which the sequence*

$$(B_n f)_{n\geq 0}$$

is orthogonal.

References

[1] J.-d. Cao, *On linear approximation methods,* (Chinese), Acta Sci. Natur. Univ. Fudan **9** (1964) 43-52.

[2] A. Lupaş, *Contribuţii la teoria aproximării prin operatori liniari,* (Romanian), Ph. D.-Thesis, Cluj-Napoca, 1976.

[3] A. Lupaş and C. Manole, *A property of Bernstein polynomials,* (Romanian), Bul. Stiintific Inst. Politehn. "Traian Vuia", Timişoara, ser. Mat.-Fiz. **26(40)** (1980) 33-36.

[4] D.D. Stancu, *On the monotonicity of the sequence formed by the first order derivatives of the Bernstein polynomials,* Math. Z., **98** (1967) 46-51.