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# On the Gamma and Beta approximation operators

Vasile Miheşan

**1.** An exponential operator is a positive linear integral operator

$$(1) \qquad (S_n f)(x) = \int_{-\infty}^{\infty} W(n,x,t)\, f(t) dt$$

whose kernel $W(n,x,t)$ satisfies the partial differential equation

$$(2) \qquad \frac{\partial W}{\partial x} = \frac{n(t-x)}{p(x)} W$$

and the normalization condition

$$(3) \qquad \int_{-\infty}^{\infty} W(n,x,t) dt = 1.$$

The exponential with $p(x)$ a quadric polinomial was studied by May, Ismail and May [1] studied exponential operators with general $p(x)$.

The exponential operators are approximation operators, that is $(S_n f)(x) \to f(x)$ as $n \to \infty$, for certain classes of functions $f$. The following are examples of exponential operators.

i) The Gauss-Weierstrass operator; $p(x) = 1,\ x \in (-\infty, \infty)$.

ii) The Szasz operator; $p(x) = x,\ x \in (0, +\infty)$.

iii) The Bernstein operator; $p(x) = x(1-x)$, $x \in (0, 1)$.

iv) The Baskakov operator; $p(x) = x(1+x)$, $x \in (0, \infty)$.

v) The Post-Widder operator; $p(x) = x^2$, $x \in (0, \infty)$.

$$(4)\qquad (P_n f)(x) = \frac{(n/x)^n}{\Gamma(n)} \int_0^\infty e^{-\frac{nt}{x}} t^{n-1} f(t)dt,\ x \in (0, \infty).$$

**2.** Let $\Gamma_p^{(\alpha,\beta)}$ the Gamma-type transform defined by [3] :

$$(5)\qquad \left(\Gamma_p^{(\alpha,\beta)} f\right)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(x e^{-\beta t} (t/p)^\alpha\right) dt$$

where $\Gamma(\cdot)$ is the Gamma function.

a) For $\alpha = 1$, $\beta = 0$ we obtain the Gamma-transform of first kind

$$(6)\quad (\Gamma_p f)(x) = \left(\Gamma_p^{(1,0)} f\right)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(\frac{xt}{p}\right) dt$$

which for $p = n$ is the Post-Widder operator.

b) For $\alpha = 0$, $\beta = 1$ we obtain the Gamma transform of second kind

$$(7)\quad \left(\Gamma_p^* f\right)(x) = \left(\Gamma_p^{(0,1)} f\right)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f(x e^{-t}) dt.$$

which has been introduced in [3].

**3.** Let $B_{p,q}^{(\alpha,\beta)}$ the Beta-type transform defined by [3]

$$B_{p,q}^{(\alpha,\beta)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(t^\alpha(1-t)^\beta\right) dt \tag{8}$$

where $B(\cdot,\cdot)$ is the Beta function.

a) For $\alpha = 1$, $\beta = 0$ we obtain the Beta-transform of first kind

$$B_{pq} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t) dt. \tag{9}$$

The corespondind Beta-operator has been introduced by Lupas [2] for $p = nx$, $q = n(1-x)$.

b) For $\alpha = 1$, $\beta = 1$ we obtain the Beta-transform of second kind which has been introduced by D.D. Stancu [4]

$$T_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} f(t)\, dt. \tag{10}$$

The coresponding Beta-operator of second kind is obtained for $p = mx$ , $q = m + 1$ [4].

**4.** Now we introduce a new beta second-kind approximating operator. We consider the Beta transform

$$\left(R_{p,q} f\right)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} f(tx) dt. \tag{11}$$

For $p=n$, $q=n+1$ we obtain [3]

$$(12) \qquad (\mathcal{B}_n f)(x) = \frac{1}{B(n,n+1)} \int_o^{\infty} \frac{t^{n-1}}{(1+t)^{2n+1}} f(tx)dt.$$

My question is if $f$ there are Gamma or Beta type operators (except the Post-Widder operator (4) ) being the exponential type operators.

## References

[1] Ismail, M.E.H., May, C.P., *On a family of approximation operators*, J. Math. Anal. Appl. 63(1978), 446-462.

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[4] Stancu, D.D., *On the Beta approximating operators of second kind*, Anal. Numer. Théor. Approx. 24 (1995), 231-239.