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On the degree of approximation by Bernstein operators

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Let $C[0,1]$ be the space of continuous functions and let $B[0,1]$ be the space of bounded functions. Let $\mathbf{N} := \{1, 2, 3, \ldots\}$ Denote by $\mathcal{P}_n, n \in \mathbf{N}$ the space of polynomials of degree at most n. Consider the Bernstein operators $B_n : B[0,1] \to \mathcal{P}_n$ defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot p_{n,k}(x),$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ f \in B[0,1],\ x \in [0,1],\ n \in \mathbf{N}.$$

For $f \in B[0,1]$ and $h > 0$, consider also the second order modulus of continuity given by

$$\omega_2(f,h) := \sup\{|f(t-\delta) - 2f(t) + f(t+\delta)|,$$

$$t \pm \delta \in [0,1] \quad , \quad 0 < \delta \leq h\}.$$

We are interested in the following type of estimates

$$(1) \qquad |B_n(f,x) - f(x)| \le C_t \cdot \omega_2\left(f,\, t\sqrt{\frac{x(1-x)}{n}}\right),$$

$$t > 0,\ f \in B[0,1],\ n \in \mathbf{N},\ x \in [0,1],$$

where C_t is a constant that depends only on t.

A first result of this type was established by H.H.Gonska [1], namely one can take $C_t = 3+t^{-2}$. Later on, the constant C_t was improved as $C_t = 1+t^{-2}$ and our paper [5], then $C_t = \frac{3}{2}+\frac{3}{4}\cdot t^{-2}$ by Gonska and Kovaceva [4] and then $C_t = 1+\frac{1}{2}\cdot t^{-2}$ in our paper [6]. All these results are cosequences of certain estimates for general positive linear operators.

Because the choice $C_t < 1$ is not possible, a special interest is given by the question if we can take $C_t = 1$. This problem is in connection to the following conjecture given by Gonska [2].

$$(2) \qquad \sup_{f \in C[0,1]\setminus \mathcal{P}_1} \sup_{n \in \mathbf{N}} \frac{\|B_n(f) - f\|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = 1.$$

The fact that this supremum is not less than 1 was observed at first by Gonska. With regard to this problem we can prove the following more precise result.

Proposition 1 For any $n \in \mathbf{N}$ and any $x \in [0,1]$

$$(3) \qquad \sup_{f \in B[0,1] \setminus \mathcal{P}_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f,\, \frac{1}{\sqrt{n}}\right)} \le 1$$

and for any $n \in \mathbf{N}$ and any $x \in [0,1] \setminus \mathbf{Q}$, where $\mathbf{Q}$ denotes the set of rational numbers, we have

$$(4) \qquad \sup_{f \in B[0,1] \setminus \mathcal{P}_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f,\, \frac{1}{\sqrt{n}}\right)} = 1.$$

A part of the result in (3) is contained in [7]. One can compare the inequality in (4) with the result obtained by H. H. Gonska and Ding Xuan-Zhou in [3] : There is a number $0 < c < 1$ (that can be explicitly given) with the property that for any $\frac{1}{2} \le a < 1$, there is an $N(a) \in \mathbf{N}$ such that for $n \ge N(a)$, $f \in C[0,1]$,

$$\sup_{1-a \le \frac{k}{n} \le a} |B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)| \le c\omega_2\left(f, \frac{1}{\sqrt{n}}\right).$$

By returning to the above stated problem we give the following

Proposition 2 There is $t > 0$ such that for any $\frac{1}{2} \le a < 1$ there is $N(a) \in \mathbf{N}$ such that for $n \in \mathbf{N}$, $n \ge N(a)$, $1 - a \le x \le a$ and $f \in B[0,1]$,

$$(5) \qquad |B_n(f,x) - f(x)| \le \omega_2\left(f,\, t \cdot \sqrt{\frac{x(1-x)}{n}}\right).$$

In Proposition 2 one can take $t = 12$.

References

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