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p-Convex set-valued maps with values in R

Dorian Popa

Abstract

For a p-convex set-valued maps with values compact subsets of $\mathbf{R}$ it is given a characterization.

Let X be a real vector space. We denote by $\mathcal{P}_0(X)$ the set of all nonempty subsets of X. A subset D of X is said to be p-convex, where p is a fixed number in the interval (0,1), if for any $x, y \in D$ we have:

$$(1-p)x + py \in D.$$

It is known (see [1]) that every p-convex and closed subset of a topological vector space is a convex set.

Let D be a p-convex and nonempty subset of X. A set-valued map $F : D \to \mathcal{P}_0(\mathbf{R})$ is said to be p-convex if for any $x, y \in D$ we have:

$$(1-p)F(x) + pF(y) \subseteq F((1-p)x + py).$$

A function $f : D \to \mathbf{R}$ is said to be p-convex (concave) if for any $x, y \in D$ we have:

$$f((1-p)x + py) \leq (\geq)(1-p)f(x) + pf(y).$$

The following assertions can be easily verified.

A set-valued map $F : D \to \mathcal{P}_0(\mathbf{R})$ is p-convex if and only if the graph of F defined by

$$\mathrm{Graph}F = \{(x, y) \in X \times \mathbf{R} : \ y \in F(x)\}$$

is a p-convex subset of the vectorial space $X \times \mathbf{R}$.

A function $f : D \to \mathbf{R}$ is p-convex if and only if the epigraph of f defined by

$$\mathrm{Epi}f = \{(x, t) \in X \times \mathbf{R} : \ f(x) \leq t\}$$

is a p-convex subset of the vector space $X \times \mathbf{R}$.

Example 1. Let $f, g : D \to \mathbf{R}$ two functions such that
$f(x) \leq g(x)$ for every $x \in D$. Then the set valued function
$F : D \to \mathcal{P}_0(\mathbf{R})$ given by the relation

$$F(x) = [f(x), g(x)]$$

for every $x \in X$ is p-convex if and only if f is p-convex and g is p-concave.

Proof. Let $x, y \in D$. We have

$$(1-p)F(x)+pF(y) = [(1-p)f(x)+pf(y), (1-p)g(x)+pg(y)]$$

and

$$F((1-p)x+py) = [f((1-p)x+py), g((1-p)x+py)].$$

The relation

$$(1-p)F(x)+pF(y) \subseteq F((1-p)x+py)$$

exists if and only if we have

$$f((1-p)x+py) \leq (1-p)f(x)+pf(y)$$

and

$$g((1-p)x+py) \leq (1-p)g(x)+pg(y),$$

so f is p-convex and g is p-concave.

Remark 1 If $F : D \to \mathcal{P}_0(\mathbf{R})$ is a p-convex set valued map with closed values, then it is convex valued.

Proof. Let $x \in D$. We have

$$(1-p)F(x)+pF(x) \subseteq F((1-p)x+px) = F(x),$$

hence $F(x)$ is a p-convex subset of $\mathbf{R}$ and being closed it is a convex subset of $\mathbf{R}$.

A characterization of midconvex set-valued maps with values in **R** is given in [2] and [3]. To obtain a characterization of p-convex set-valued map with compact values in $\mathcal{P}_0(\mathbf{R})$ we need some lemmas.

Lemma 1 [1] *Let $p \in (0,1)$. Denote by $(P_n)_{n\geq 1}$ the sequence of sets defined as follows:*

$$P_1 = \{0, p, 1\};$$

if

$$P_n = \{0, p_n^{(1)}, \ldots, p_n^{(2^n)}, 1\},$$

where

$$0 < p_n^{(1)} < \ldots < p_n^{(2^n)} < 1,$$

is defined, put

$$P_{n+1} = P_n \cup \{(1-p)p_n^{(k-1)} + pp_n^{(k)} :\ 1 \leq k \leq 2^n + 1\}$$

where $p_0^{(0)} = 0$ and $p_n^{(2^n+1)} = 1$. Then the set

$$P = \bigcup_{n\geq 1} P_n$$

is dense in the interval $[0,1]$.

Lemma 2 ([1]) *Let X be a real vector space and D a p-convex and nonempty subset of X. Then D is q-convex for each $q \in P$,* *where P is the set defined in Lemma 1.*

Lemma 3 *Let X be a real vector space, D a p-convex and nonempty subset of X. If a set-valued map $F : D \to \mathcal{P}_0(Y)$ is p-convex then it is q-convex for any $q \in P$, where P is the set defined in Lemma 1.*

Proof. From the p-convexity of F it results that Graph$F$ is a p-convex set of $X \times \mathbf{R}$, and using Lemma 2 we obtain that Graph$F$ is q-convex for any $q \in P$. Then F is q-convex for any $q \in P$.

Theorem 1 *Let X be a real space and $F : X \to \mathcal{P}_0(\mathbf{R})$ a p-convex set-valued map with compact values. Then there exists an additive function $a : X \to \mathbf{R}$ and a compact interval I in $\mathbf{R}$ such that for every $x \in X$ we have:*

$$F(x) = a(x) + I.$$

Proof. For any $x \in X$ let

$$f(x) = \inf F(x) \quad \text{and} \quad g(x) = \sup F(x)\ .$$

Then $f : X \to \mathbf{R}$ is p-convex and $g : X \to \mathbf{R}$ is p-concave. Indeed for every $x, y \in X$ we have:

$$f((1-p)x+py) = \inf F((1-p)x+py) \le \inf((1-p)F(x)+pF(y)) =$$

$$= \inf((1-p)F(x) + \inf(pF(y)) = (1-p)f(x) + pf(y)$$

hence f is a p-convex function and analogously g is a p-concave function. We have also $f(x) \leq g(x)$ for any $x \in X$.

Let $h : X \to \mathbf{R}$, $h(x) = g(x) - f(x)$ for any $x \in X$. Obviously h is p-concave and $h(x) \geq 0$ for every $x \in X$. We prove that h is a constant function.

The function $-h$ is p-convex, hence the set $\mathrm{Epi}(-h)$ is p-convex and it follows from Lemma 2 that $\mathrm{Epi}(-h)$ is q-convex for every $q \in P$. It results that $-h$ is a q-convex function for $q \in P$, and forward h is q-concave for $q \in P$.

Suppose that h is nonconstant. Then there exists $x, y \in X$, $x \neq y$, so that $h(x) < h(y)$. Using the density of P in [0,1] it results that there exists a $t > 1$, $\frac{1}{t} \in P$ such that:

$$t(h(x) - h(y)) + h(y) < 0.$$

From the q-concavity of h with $q \in P$ we have:

$$h(x) = h\left(\frac{1}{t}(tx + (1-t)y) + \left(1 - \frac{1}{t}\right) y\right) \geq$$

$$\geq \frac{1}{t} h(tx + (1-t)y) + \left(1 - \frac{1}{t}\right) h(y)$$

and foreward it results

$$h(tx+(1-t)y) \leq th(x)+(1-t)h(y) = t(h(x)-h(y))+h(y) < 0$$

contradiction with nonnegativity of the values of h.

It results that there exists $c \in \mathbf{R}$ such that $h(x) = c$, for any $x \in X$.

The function $f = g - c$ is p-concave and being p-convex satisfie the relation

$$f((1-p)x + py) = (1-p)f(x) + pf(y). \tag{1}$$

There exists an additive function $a : X \to \mathbf{R}$ and $k \in \mathbf{R}$ such that $f(x) = a(x) + k$ for any $x \in X$.

For $x = 0$ and $y \in X$ in (1) we have

$$f(py) = pf(y) + (1-p)f(0). \tag{2}$$

For $y = 0$ and $x \in X$ in (1) we have

$$f((1-p)x) = (1-p)f(x) + pf(0). \tag{3}$$

Let $u, v \in X$. From (1), (2), (3) we have

$$f(u+v)= f\left((1-p)\frac{u}{1-p}+p\frac{v}{p}\right) =$$

$$= (1-p)f\left(\frac{u}{1-p}\right)+pf\left(\frac{v}{p}\right) =$$

$$= (1-p)f\left(\frac{u}{1-p}\right)+pf(0)+pf\left(\frac{v}{p}\right)+$$

$$+(1-p)f(0)-(1-p)f(0)-pf(0) =$$

$$= f(u)+f(v)-f(0).$$

The function $a : X \to \mathbf{R}$, $a(x) = f(x) - f(0)$ is additive. Indeed for any $x, y \in X$ we have:

$$a(x+y) = f(x+y)-f(0) = f(x)+f(y)-f(0)-f(0) = a(x)+a(y).$$

Denote $k = f(0)$; we obtain $f(x) = a(x) + k$ for any $x \in X$.

Let $x \in X$. We have taking account of the convexity of values F that

$$F(x) = [f(x), g(x)] = [a(x)+k, a(x)+k+c] =$$

$$= a(x)+[k, k+c] = a(x)+I \ ,$$

where $I = [k, k+c]$.

Remark 2 If p is a rational number in [0,1] then the converse of Theorem 1 is true.

Proof. Let $a : X \to \mathbf{R}$ an additive function, I a compact interval in $\mathbf{R}$ and $F : X \to \mathcal{P}_0(\mathbf{R})$, $F(x) = a(x) + I$ for every $x \in X$. Taking account that a is rationaly homogeneous it results that

$$F((1-p)x + py) = a((1-p)x + py) + I =$$

$$= (1-p)a(x) + pa(y) + (1-p)I + pI =$$

$$= (1-p)F(x) + pF(y)$$

for every $x, y \in X$.

References

[1] A. Aleman, *On some generalizations of convex sets and convex functions*, Mathematica - Révue d'analyse numérique et de la théorie de l'approximation, Tome 14, No.1(1985), 1-6.

[2] K. Nikodem, *Midpoint convex functions majorized by midpoint concave functions*, Aequationes Math. 32(1987), 45-51.

[3] Z. Kominek, *On a problem of K. Nikodem*, Arch. Math. 50(1988), 287-288.