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The evaluation of the remainders in approximation formulas by linear positive operators of interpolatory type

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Let $f \in C[0,1]$ and

$$p_{m,k}^{<\alpha>}(u,v) = \binom{m}{k} \frac{u^{[k,-\alpha]}v^{[m-k,-\alpha]}}{(u+v)^{[m,-\alpha]}} ,$$

where $(u,v) \in [0,1]\times[0,1]$ and $\alpha \in \mathbf{R}_+$.

In 1968 we have introduced and investigated in [2] a Bernstein type polynomial $S_m^{<\alpha>}f$, defined by

$$\left(S_m^{<\alpha>}f\right)(x) = \sum_{k=0}^{m} p_{m,k}^{<\alpha>}(x,1-x) f\left(\frac{k}{n}\right) .$$

For the remainder of the corresponding approximation formula we have obtained the following representation (see [4], [6]):

$$(2) \qquad (R_m^{<\alpha>}f)(x) =$$

$$= -\frac{x(1-x)}{m}\frac{1+\alpha m}{1+\alpha}\sum_{k=0}^{m-1} p_{m-1,k}^{<\alpha>}(x+\alpha, 1-x+\alpha)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right].$$

For $\alpha = 0$ we obtain an expression (given in 1964 in our paper [1]) for the remainder of Bernstein's approximation formula.

In the case of a generalized Meyer-König and Zeller operator $M_m^{<\alpha>}$ (see [3]) we have obtained in [5] and [7] the following expression for the corresponding remainder

$$(R_m^{<\alpha>} f)(x) = \tag{3}$$

$$= -x(1-x) \sum_{k=0}^{\infty} (m+1+k)^{-1} w_{m,k}^{<\alpha>}(x+\alpha, 1-x+\alpha) \left[x, \frac{k}{m+k}, \frac{k+1}{m+k+1}; f \right],$$

where

$$w_{m,k}^{<\alpha>}(u,v) = \binom{m+k}{k} \frac{u^{[k,-\alpha]} v^{[m+1,-\alpha]}}{(u+v)^{[m+1+k,-\alpha]}} \ .$$

Problem 1 *We propose to find an expression similar with (2) and (3) for the remainder of approximation formula of a function $f \in C[0,1]$ by means of an operator of Bernstein type of Cheney-Sharma, defined by the formula*

$$(Q_m f)(x;\beta) = \sum_{k=0}^{m} p_{m,k}(x;\beta) f\left(\frac{k}{m}\right) \quad ,$$

where

$$\beta = o\left(\frac{1}{m}\right)$$

and

$$p_{m,k}(x;\beta) = \binom{m}{k} \frac{x(x+k\beta)^{k-1}(1-x)[1-x+(m-k)\beta]^{m-1-k}}{(1+m\beta)^{m-1}} \quad ,$$

which has also the degree of exactness equal with one.

Problem 2 *How can be obtained an evaluation of the order of approximation of a function $f \in C[0,1]$ by means of the Bernstein polynomials, using the expression of the remainder given in [1], in terms of the second order modulus of smoothness?*

References

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