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A convexity problem for a special function

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Consider

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$$

the Γ Euler's function, $x > 0$.

Let's take the following case:

"Show that the function $x \mapsto f(x) = (\Gamma(x+1))^{\frac{1}{x}}$, $x > 0$, is strictly concave."

In the case of the natural variable $x = n$, the concave character of the sequence $(a_n)_{n\geq 1}$ of general term

$$a_n = (\Gamma(n+1))^{\frac{1}{n}}$$

(i.e. $a_{n+1} > \dfrac{1}{2}(a_n + a_{n+2})$) would lead us to obtain the monotony of the T.Lalescu's sequence

$$(L_n)_{n\geq 1} \quad , \quad L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \quad , \quad L_n > L_{n+1}.$$

This sequence has been extensively investigated beginning with Tiberiu Popoviciu. But the problems of the monotony of the sequence $(L_n)_{n\geq 1}$ was raised for the first time by Professor Alexandru Lupaş (see [3]). A remarkable theorem of H. Bohr and I. Mollerup asserts that there is a unique function $F\colon (0,\infty) \to (0,\infty)$, logarithmically convexe, with $F(x+1) = xF(x)$ and $F(1) = 1$; that is $F = \Gamma$ (see [1], [2]). To establish the inequality $f''(x) < 0$ we should take into account the following inequalities

$$0 < \ln\Gamma(x) - \left[\left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}(\ln 2\pi)\right] < \frac{1}{x} \quad , \quad (x > 1),$$

$$\frac{1}{2x} < \ln x - \frac{\Gamma'(x)}{\Gamma(x)} < \frac{1}{x} \quad , \quad (x > 1),$$

(see [5], pag. 288). This method hasn't been sufficiently sought into. We also note that the function

$$x \mapsto x\frac{(\Gamma(x+1))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}$$

is logarithmically concave (see [5], pag. 289, [4]).

References

[1] Bohr, H., Mollerup, I.: *Loerbog I matematisk Analyse*, t. III, Kopenhagen, 1922.

[2] Lazarevič, I., Lupaş, A.: *An inequality for the gamma function*, G.M. (seria metodică), nr. 1 – 2/1982, pag. 82 – 84.

[3] Lupaş, A.: *Probleme deschise*, G.M., nr. 1/1979, pag. 33.

[4] Minc, H., Sathre, L.: *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. (2), 14, (1964/65), pag. 41 – 46.

[5] Mitrinovič, D.S. (in cooperation with Vasič P.M.): *Analytic Inequalities*, Springer Verlag, Berlin, Band 165.

[6] Popoviciu, T.: *Asupra calculului unei limite*, G.M. (seria A), vol. LXXVI, nr. 1/1971.