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Some considerations concerning the rotary motion of a rigid body around the mass center

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Abstract

We make a presentation of the tangent fibre bundle of the group of the orthogonal transforms and, on this fibre bundle, we construct a riemannian structure whose metric is defined by the intermedium of the kinetic energy of rotation.

Corresponding to this structure we calculate the Cristoffels symbols of second kind to obtain the equations of the geodesics and we look for an orthogonal basis in this structure such that the equations of the geodesics should coincide with the Eulers equations.

We identify these three directions with the central axes of the rigid body and we show that the movement of these three directions cannot be considered with respect to an absolute frame in the given structure of the riemannian variety. Consequently the frame that we chose to represent the central axes of the rigid body has a continuous movement, ((47),2), the angular velocity passing, periodically, through the same values but with respect to a moving frame called frame "of observations".

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1 General considerations

We consider a rigid body whose principal axes will be denoted by Cx_1', Cx_2', Cx_3' (the frame of reference R) while i_1^2, i_2^2, i_3^2 represent the central inertial moments.

Let $(C; x, y, z)$ be an orthogonal frame R and let denote by $\psi.\theta, \varphi$ the Eulers angles defining the position of R' with respect to R; Fig.1.

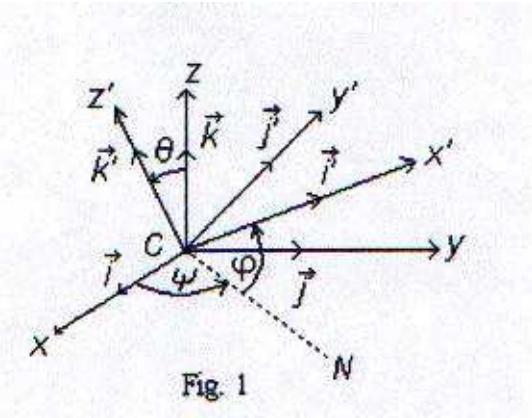

Fig. 1

Between the unit vectors of the two frames we have

$$\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = A \begin{pmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{pmatrix} \tag{1}$$

where

$$A(\psi,\theta,\varphi) = \tag{2}$$

$$= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we consider another orthogonal frame R_1, $(C; x_1, y_1, z_1)$ and we denote by ψ_1, θ_1, φ_1 the Eulers angles between the frame R' and R_1, respectively, by ψ_0, θ_0, φ_0 the Eulers angles between the frame R and R_1, then between the corresponding matrices we have

$$A(\psi_1,\theta_1,\varphi_1) = A(\psi_0,\theta_0,\varphi_0)A(\psi,\theta,\varphi) \tag{3}$$

From (3) we have the rules of composition for the group of parameters of the orthogonal transformations SO_3.

The orthogonal invariant at right, ω, of this group of transformations has the components

$$\omega = \begin{pmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{pmatrix} = \tag{4}$$

$$A^{-1}(\psi_1,\theta_1,\varphi_1)A(\psi_1,\theta_1,\varphi_1) = A^{-1}(\psi,\theta,\varphi)A(\psi,\theta,\varphi)$$

where

$$(5)\begin{cases}\omega_1' = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi = \dot{\psi}_1\sin\theta_1\sin\varphi_1 + \dot{\theta}_1\cos\varphi_1\\ \omega_2' = \dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi = \dot{\psi}_1\sin\theta_1\cos\varphi_1 - \dot{\theta}_1\sin\varphi_1\\ \omega_3' = \dot{\psi}\cos\theta + \dot{\varphi} = \dot{\psi}_1\cos\theta_1 + \dot{\varphi}_1\end{cases}$$

If we consider an orthogonal transformation at left corresponding to the constant matrix $A(\psi_0, \theta_0, \varphi_0)$ in (3) what means that instead of the central axes we consider a jointly with them frame, then between the orthogonal invariants of the two frames we have

$$(6)\qquad \begin{pmatrix} 0 & \widetilde{\omega_3'} & -\widetilde{\omega_2'}\\ -\widetilde{\omega_3'} & 0 & \widetilde{\omega_1'}\\ \widetilde{\omega_2'} & -\widetilde{\omega_1'} & 0\end{pmatrix} =$$

$$= A^{-1}(\psi_0, \theta_0, \varphi_0)\begin{pmatrix} 0 & \widetilde{\omega_3'} & -\widetilde{\omega_2'}\\ -\widetilde{\omega_3'} & 0 & \widetilde{\omega_1'}\\ \widetilde{\omega_2'} & -\widetilde{\omega_1'} & 0\end{pmatrix} A(\psi_0, \theta_0, \varphi_0)$$

equation equivalent to

$$\begin{pmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\end{pmatrix} = A(\psi_0, \theta_0, \varphi_0)\begin{pmatrix}\widetilde{\omega_1'}\\ \widetilde{\omega_2'}\\ \widetilde{\omega_3'}\end{pmatrix}$$

where, $\widetilde{\omega_1'}, \widetilde{\omega_2'}, \widetilde{\omega_3'}$, are the components of the orthogonal invariant as functions on the angles $\psi_1, \theta_1, \varphi_1$.

In the frame R the components of the kinetic moment calculated with respect to the mass center are

$$(7)\qquad \begin{cases} k^1 = i_1^2(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi) \\ k^2 = i_1^2(\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi) \\ k^3 = i_2^3(\dot{\psi}\cos\theta + \dot{\varphi}) \end{cases}$$

The rotary kinetic energy around the mass center is given by

$$\begin{aligned} E^{rot} &= \frac{1}{2}(i_1^2\omega_1^2 + i_2^2\omega_2^2 + i_3^2\omega_3^2) = \\ &= \frac{1}{2}\{[(i_1^2\sin^2\varphi + i_2^2\cos^2\varphi)\sin^2\theta + i_3^2\sin^2\varphi]\psi^2 + \\ &+(i_1^2\cos^2\varphi + i_2^2\sin^2\varphi)\dot{\theta}^2 + i_3^2\dot{\varphi}^2 + 2i_3^2\dot{\psi}\dot{\varphi}\cos\theta + \\ &+(i_1^2 - i_2^2)\dot{\psi}\dot{\theta}\sin\theta\sin\varphi\cos\varphi\} \end{aligned} \qquad (8)$$

We will also denote with V_3 the riemannian variety of the group of parameters ψ, θ, φ of the orthogonal transformations SO_3 endowed with the metric

$$
\begin{aligned}
ds^2 &= [\sin^2\theta(i_1^2\sin^2\varphi + i_2^2\cos^2\varphi) + i_3^2\cos^2\theta]d\psi^2 + \\
&+(i_1^2\cos^2\varphi + i_2^2\sin^2\varphi)d\theta^2 + i_3^2 d\varphi^2 + \\
&+2(i_1^2 - i_2^2)\sin\theta\sin\varphi\cos\varphi d\psi d\omega + 2i_3^2\cos\vartheta d\psi d\varphi.
\end{aligned} \tag{9}
$$

Considerations concerning the time measuring

The newtonian way of the time measuring is given by the principle of inertia from whose enunciation we retain that "any body preserves its state of uniform and rectilinear translation when upon it acts no force at all".

Consequently, to define the uniform lapse of the newtonian time in the absolute frame, we divide the rectilinear trajectory covered by the body upon whom acts no force at all in equal portions and we choose for the time measuring an apparatus (pulsator) which emits the same number of pulsations Δt on each portion Δs covered by the body. So we have $\Delta s = c\Delta t$ and the apparatus for the time measuring is correct from point of view of the principle of inertia.

Let introduce now the notion of "admissible" time or "Lagrangian" time associated to a rotary motion of a body as follows.

We will consider, for the time measuring of the "admissible" time associated to a "body that rotates around its mass center because of the initial conditions and upon whom acts no force or couple at all", that division of the time axis which preserves the kinetic energy of the body as constant.

From (8) and (9) we have

$$ds^2 = 2E^{rot}dt^2 \ . \tag{10}$$

We remark that it is not obligatory that the "admissible" time coincides with the newtonian time.

So, if for the newtonian time $E^{rot} = \frac{1}{2}Ce^{-2t'}$ would be constant, for the "admissible" time we will have $dt = e^{-t'}dt'$.

For ds given by (9) or (10) the "admissible" time is definite until an affine transform on the affine arc of the autoparallel curve.

The geometric structure of the variety V_3

The metric tensor of V_3, given by (9) is

$$(11)\qquad \begin{cases} g_{11} = \sin^2\theta(i_1^2\sin^2\varphi + i_2^2\cos^2\varphi + i_3^2\cos^2\theta)\ , \\ g_{22} = i_1^2\cos^2\varphi + i_2^2\sin^2\varphi\ , \\ g_{12} = (i_1^2 - i_2^2)\sin\theta\sin\varphi\cos\varphi\ , \\ g_{13} = i_3^2\cos\theta\ , \\ g_{23} = 0\ , \\ g_{33} = i_3^2\ . \end{cases}$$

Let denote by $\dfrac{\partial}{\partial\theta^1}, \dfrac{\partial}{\partial\theta^2}, \dfrac{\partial}{\partial\theta^3}$ the natural basis of V_3 satisfying the equations

$$(12)\qquad \frac{\partial}{\partial\theta^i}\cdot\frac{\partial}{\partial\theta^j} = g_{ij}\ .$$

The Cristoffels symbols of second kind calculated accordingly to the known formula

$$\left|\begin{matrix} h \\ ij \end{matrix}\right| = g^{hk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

are ([1],(12), for $\theta \neq 0$)

$$
(13)\quad \begin{cases}
\left|\begin{matrix}1\\11\end{matrix}\right| = \dfrac{(i_2^2-i_1^2)(i_3^2-i_1^2-i_2^2)}{i_1^2 i_2^2}\sin\varphi\cos\varphi\cos\theta,\\
\left|\begin{matrix}1\\22\end{matrix}\right| = 0,\ \left|\begin{matrix}1\\33\end{matrix}\right| = \dfrac{(i_2^2-i_1^2)(i_3^2-i_1^2-i_2^2)}{i_1^2 i_2^2}\sin\varphi\cos\varphi\cos\theta,\\
\left|\begin{matrix}1\\13\end{matrix}\right| = \dfrac{(i_1^2-i_2^2)(i_1^2+i_2^2-i_3^2)}{2i_1^2 i_2^2}\sin\varphi\cos\varphi,\\
\left|\begin{matrix}1\\23\end{matrix}\right| = \dfrac{-i_1^2(i_2^2+i_2^2-i_3^2)\cos^2\varphi - i_2^2(i_1^2+i_3^2-i_2^2)\sin^2\varphi}{2i_1^2 i_2^2\sin\theta}\\
\left|\begin{matrix}2\\22\end{matrix}\right| = 0\ ,\ \left|\begin{matrix}2\\33\end{matrix}\right| = 0,\\
\left|\begin{matrix}2\\11\end{matrix}\right| = \dfrac{\sin\theta\cos\theta}{i_1^2 i_2^2}[(i_3^2-i_1^2)i_1^2\sin^2\varphi + (i_3^2-i_2^2)\cos^2\varphi],\\
\left|\begin{matrix}2\\12\end{matrix}\right| = \dfrac{(i_2^2-i_1^2)(i_1^2+i_2^2-i_3^2)}{2i_1^2 i_2^2}\sin\varphi\cos\varphi\cos\theta,\\
\left|\begin{matrix}2\\13\end{matrix}\right| = \dfrac{\sin\theta}{2i_1^2 i_2^2}[i_1^2(i_3^2+i_2^2-i_2^2)\sin^2\varphi + i_2^2(i_3^2+i_1^2-i_2^2)\cos^2\varphi],\\
\left|\begin{matrix}2\\23\end{matrix}\right| = \dfrac{\sin\varphi\cos\varphi}{2i_1^2 i_2^2}(i_2^2-i_1^2)(i_1^2+i_2^2-i_3^2),\\
\left|\begin{matrix}3\\11\end{matrix}\right| = \dfrac{i_1^2-i_2^2}{i_1^2 i_2^2 i_3^2}[i_3^2(i_3^2-i_1^2-i_2^2)\cos^2\theta - i_1^2 i_2^2\sin^2\theta]\sin\varphi\cos\varphi,\\
\left|\begin{matrix}3\\22\end{matrix}\right| = \dfrac{i_1^2-i_2^2}{i_3^2}\sin\varphi\cos\varphi,\\
\left|\begin{matrix}3\\12\end{matrix}\right| = \dfrac{\sin\theta}{2i_3^2}[(i_2^2-i_1^2)(\cos^2\varphi-\sin^2\varphi) - i_3^2] +\\
\qquad + \dfrac{(i_3^2-i_1^2-i_2^2)\cos^2\theta}{2i_1^2 i_2^2\sin^2\theta}(i_1^2\cos^2\varphi + i_2^2\sin^2\varphi),
\end{cases}
$$

$$\begin{cases} \left|\begin{matrix}3\\23\end{matrix}\right| = \dfrac{\cos\theta}{2i_1^2 i_2^2 \sin\theta}[i_1^2(i_3^2+i_2^2-i_1^2)\cos^2\varphi + i_2^2(i_1^2+i_3^2-i_2^2)\sin^2\varphi], \\ \left|\begin{matrix}3\\13\end{matrix}\right| = \dfrac{(i_1^2-i_2^2)(i_3^2+i_1^2-i_2^2)}{2i_1^2 i_2^2}\sin\varphi\cos\varphi\cos\theta, \ \left|\begin{matrix}3\\33\end{matrix}\right| = 0. \end{cases}$$

The equations of the geodesics (given in [1], (15)) are:

$$(14)\begin{cases} \dfrac{d^2\psi}{ds^2} + \dfrac{(i_2^2-i_1^2)(i_3^2-i_1^2-i_2^2)}{i_1^2 i_2^2}\sin\varphi\cos\varphi\cos\theta\left(\dfrac{d\psi}{ds}\right)^2 + \\ \quad + \dfrac{i_1^2-i_2^2-i_3^2}{i_1^2 i_2^2}(i_1^2\cos^2\varphi + i_2^2\sin^2\varphi)\dfrac{\cos\theta}{\sin\theta}\dfrac{d\psi}{ds}\dfrac{d\theta}{ds} + \\ \quad + \dfrac{1}{i_1^2 i_2^2}[i_1^2(i_1^2-i_2^2-i_3^2)\cos^2\varphi + \\ \quad + i_2^2(i_2^2-i_1^2-i_3^2)\sin^2\varphi]\dfrac{1}{\sin\theta}\dfrac{d\theta}{ds}\dfrac{d\varphi}{ds} + \\ \quad + \dfrac{(i_2^2-i_1^2)(i_1^2-i_2^2-i_3^2)}{i_1^2 i_2^2}\sin\varphi\cos\varphi\dfrac{d\psi}{ds}\dfrac{d\varphi}{ds} = 0, \\ \dfrac{d^2\theta}{ds^2} + \dfrac{i_1^2(i_3^2-i_1^2)\sin^2\varphi + i_2^2(i_3^2-i_2^2)\cos^2\varphi}{i_1^2 i_2^2}\sin\theta\cos\theta\left(\dfrac{d\psi}{ds}\right)^2 + \\ \quad + \dfrac{i_2^2-i_1^2}{i_1^2 i_2^2}(i_1^2+i_2^2-i_3^2)\sin\varphi\cos\varphi\cos\theta\dfrac{d\psi}{ds}\dfrac{d\theta}{ds} + \\ \quad + \dfrac{\sin\theta}{i_1^2 i_2^2}[i_1^2(i_3^2+i_2^2-i_1^2)\cos^2\varphi + \\ \quad + i_2^2(i_3^2+i_1^2-i_2^2)\sin^2\varphi]\dfrac{d\psi}{ds}\dfrac{d\varphi}{ds} + \\ \quad + \dfrac{(i_2^2-i_1^2)(i_1^2+i_2^2-i_3^2)}{i_1^2 i_2^2}\sin\varphi\cos\varphi\dfrac{d\theta}{ds}\dfrac{d\varphi}{ds} = 0, \end{cases}$$

$$\left\{\begin{array}{l} \dfrac{d^2\varphi}{ds^2}+\dfrac{i_1^2-i_2^2}{i_1^2i_2^2}\sin\varphi\cos\varphi[(i_3^2-i_1^2-i_2^2)\cos^2\theta- \\ \quad -\dfrac{i_1^2i_2^2}{i_3^2}\sin^2\theta]\left(\dfrac{d\psi}{ds}\right)^2+\dfrac{i_1^2-i_2^2}{i_3^2}\sin\varphi\cos\varphi\left(\dfrac{d\theta}{ds}\right)^2+ \\ \quad +\left\{\dfrac{\sin\theta}{i_3^2}[(i_2^2-i_1^2)\cos^2\varphi-i_3^2]+\right. \\ \quad \left.+\dfrac{i_3^2-i_1^2-i_2^2}{i_1^2i_2^2}(i_1^2\cos^2\varphi+i_2^2\sin^2\varphi)\dfrac{\cos^2\theta}{\sin\theta}\right\}\dfrac{d\psi}{ds}\dfrac{d\theta}{ds}+ \\ \quad +\dfrac{i_3^2-i_1^2-i_2^2}{i_1^2i_2^2}(i_1^2-i_2^2)\sin\varphi\cos\varphi\cos\theta\dfrac{d\psi}{ds}\dfrac{d\theta}{ds}+ \\ \quad +\dfrac{\cos\theta}{i_1^2i_2^2\sin\theta}[i_1^2(i_3^2+i_2^2-i_1^2)\cos^2\theta+ \\ \quad +i_2^2(i_3^2+i_1^2-i_2^2)\sin^2\varphi]\dfrac{d\theta}{ds}\dfrac{d\varphi}{ds}=0. \end{array}\right.$$

In the particular case, $\theta = 0$, to write the geodesics we use the formula

$$g_{ij}\frac{d^2x^j}{ds^2}+|jk,i|\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

where $|jk,i|$ are the symbols of Cristoffel of first kind:

$$|jk,i|=\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^i}\right)\ .$$

So we will obtain an equation only

$$\frac{d^2}{ds^2}(\psi+\varphi)=0\ . \tag{15}$$

Let consider as invariant the following one-forms

$$(16)\qquad \begin{cases} d\eta^1 = i_1^2(\sin\theta\sin\varphi d\psi - \cos\varphi d\theta) \\ d\eta^2 = i_2^2(\sin\theta\cos\varphi d\psi - \sin\varphi d\theta) \\ d\eta^3 = i_3^2(\cos\theta d\psi + d\varphi). \end{cases}$$

The values of these one-forms on the tangent vector

$$\vec{k} = \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\psi}\frac{\partial}{\partial\psi} + \dot{\varphi}\frac{\partial}{\partial\varphi}$$

represent the components of the angular moment $d\eta^1(\vec{k}) = k^1$, $d\eta^2(\vec{k}) = k^2$, $d\eta^3(\vec{k}) = k^3$ (see the equations (7)).

We will denote by $\dfrac{\partial}{\partial\eta^1}, \dfrac{\partial}{\partial\eta^2}, \dfrac{\partial}{\partial\eta^3}$ the covariant vectors for which $d\eta^1$, $d\eta^2$, $d\eta^3$ represent a dual basis

$$(17)\qquad d\eta^i\left(\frac{\partial}{\partial\eta^1}\right) = \delta_j^i\ .$$

With respect to the natural basis the expressions of $\dfrac{\partial}{\partial\eta^j}$ defined in (17), are

$$(18)\qquad \frac{\partial}{\partial\eta^j} = \mu_j^i\frac{\partial}{\partial\theta^i}$$

where

$$(19)\qquad [\mu_j^i] = \begin{pmatrix} \dfrac{\sin\varphi}{i_1^2\sin\theta} & \dfrac{\cos\varphi}{i_1^2} & -\dfrac{\sin\varphi\cos\theta}{i_1^2\sin\theta} \\ \dfrac{\cos\varphi}{i_2^2\sin\theta} & -\dfrac{\sin\varphi}{i_2^2} & -\dfrac{\cos\varphi\cos\theta}{i_2^2\sin\theta} \\ 0 & 0 & \dfrac{1}{i_3^2} \end{pmatrix}$$

respectively

$$\frac{\partial}{\partial\theta^i}=\lambda_i^j\frac{\partial}{\partial\eta^j} \tag{20}$$

where

$$[\lambda_i^j]=\begin{pmatrix} i_1^2\sin\theta\sin\varphi & i_2^2\sin\theta\cos\varphi & i_3^2\cos\theta \\ i_1^2\cos\varphi & -i_2^2\sin\varphi & 0 \\ 0 & 0 & i_3^2 \end{pmatrix} \tag{21}$$

A vector from the tangent space to V_3 and having the components with respect to the natural basis, with respect to the basis $\frac{\partial}{\partial\eta^1},\frac{\partial}{\partial\eta^2},\frac{\partial}{\partial\eta^3}$, will have the components of the angular moment:

$$\dot\theta\frac{\partial}{\partial\theta}+\dot\psi\frac{\partial}{\partial\psi}+\dot\varphi\frac{\partial}{\partial\varphi}=k^1\frac{\partial}{\partial\eta^1}+k^2\frac{\partial}{\partial\eta^2}+k^3\frac{\partial}{\partial\eta^3}\ .$$

The above relation represents the reason of the change of basis (18). If we calculate the components of the metric tensor g_{ij} in the basis $\frac{\partial}{\partial\eta^1},\frac{\partial}{\partial\eta^2},\frac{\partial}{\partial\eta^3}$, using the formula $g_{kl}=\mu_k^i\mu_l^j g_{ij}$, we find

$$[g_{kl}]=\begin{pmatrix} \frac{1}{i_1^2} & 0 & 0 \\ 0 & \frac{1}{i_2^2} & 0 \\ 0 & 0 & \frac{1}{i_1^2} \end{pmatrix} \tag{22}$$

whence the fact that $\frac{\partial}{\partial\eta^1},\frac{\partial}{\partial\eta^2},\frac{\partial}{\partial\eta^3}$ is also an orthogonal basis.

The connection coefficients in this basis are calculated with the formula

$$\gamma_{bc}^{a}=\left[\left|\begin{matrix} i \\ jk \end{matrix}\right| \gamma_{i}^{a}-\frac{\partial\gamma_{r}^{a}}{\partial\theta^{j}}\right]\mu_{b}^{r}\mu_{c}^{j}$$

which assures the preservation of the parallelism of the riemannian structure. We obtain

$$(23)\qquad \begin{cases} \gamma_{23}^{1}=\dfrac{i_2^2-i_1^2-i_2^2}{2i_2^2i_3^2}\ ,\ \gamma_{31}^{2}=\dfrac{i_1^2-i_2^2-i_3^2}{2i_1^2i_3^2}\ , \\ \gamma_{12}^{3}=\dfrac{i_2^2-i_3^2-i_1^2}{2i_1^2i_2^2}\ ,\ \gamma_{32}^{1}=\dfrac{i_3^2+i_1^2-i_2^2}{2i_2^2i_3^2}\ , \\ \gamma_{13}^{2}=\dfrac{i_1^2+i_2^2-i_3^2}{2i_1^2i_3^2}\ ,\ \gamma_{21}^{3}=\dfrac{i_2^2+i_3^2-i_1^2}{2i_1^2i_2^2}\ , \end{cases}$$

the other coefficients being zero.

The equations of the geodesics (14) become the Eulers equations.

Let normalize the basis $\dfrac{\partial}{\partial\eta^1},\dfrac{\partial}{\partial\eta^2},\dfrac{\partial}{\partial\eta^3}$ and let denote by

$$(24)\qquad \vec{e}_1'=i_1\frac{\partial}{\partial\eta^1}\quad,\quad \vec{e}_2'=i_2\frac{\partial}{\partial\eta^2}\quad,\quad \vec{e}_3'=i_3\frac{\partial}{\partial\eta^3}$$

respectively

$$(25)\qquad dx^1=\frac{1}{i_1}d\eta^1\quad,\quad dx^2=\frac{1}{i_2}d\eta^2\quad,\quad dx^3=\frac{1}{i_3}d\eta^3$$

the corresponding dual forms. In the new basis the metric tensor g_{ij} becomes $g_{\alpha\beta}=\dfrac{1}{i_\alpha i_\beta}\delta_\alpha^i\delta_\beta^j g_{ij}=\delta_{\alpha\beta}$ and, consequently, the frame defined by $\vec{e}_1'$, $\vec{e}_2'$, $\vec{e}_3'$ is an orthogonal frame.

The connection coefficients corresponding to these three directions are

$$(26)\qquad \begin{cases} \gamma_{23}^{1}=\dfrac{i_3^2-i_1^2-i_2^2}{2i_1i_2i_3}\ ,\ \gamma_{31}^{2}=\dfrac{i_1^2-i_2^2-i_3^2}{2i_1i_2i_3}\ , \\ \gamma_{12}^{3}=\dfrac{i_2^2-i_3^2-i_1^2}{2i_1i_2i_3}\ ,\ \gamma_{32}^{1}=\dfrac{i_3^2+i_1^2-i_2^2}{2i_1i_2i_3}\ , \\ \gamma_{13}^{2}=\dfrac{i_1^2+i_2^2-i_3^2}{2i_1i_2i_3}\ ,\ \gamma_{21}^{3}=\dfrac{i_2^2-i_3^2-i_1^2}{2i_1i_2i_3}\ , \end{cases}$$

We will calculate on the system of orthogonal congruencies (25) the Riccis rotation coefficients which are the only connection coefficients which preserve the system of orthogonal congruencies.

We use the formulas, [3],

$$(27)\qquad \gamma_{bc}^{a}=-\frac{1}{2}(w_{bc}^{a}+w_{ca}^{b}+w_{ba}^{c})$$

where

$$(28)\qquad w_{bc}^{a}=\left[\frac{\partial\lambda_i^a}{\partial\theta^j}-\frac{\partial\lambda_j^a}{\partial\theta^i}\right]\mu_b^i\mu_c^j$$

and μ_a^i are the parameters of the orthogonal congruencies (29) respectively λ_i^a are the moments of these three congruencies.

From (19), (21) and (25) we have

$$
(29)\qquad
\begin{aligned}
[\lambda_i^a] &= \begin{pmatrix} i_1 \sin\theta \sin\varphi & i_1 \cos\varphi & 0 \\ i_2 \sin\theta \cos\varphi & -i_2 \sin\varphi & 0 \\ i_3 \cos\theta & 0 & i_3 \end{pmatrix}, \\
[\mu_b^i] &= \begin{pmatrix} \dfrac{\sin\varphi}{i_1 \sin\theta} & \dfrac{\cos\varphi}{i_2 \sin\theta} & 0 \\ \dfrac{\cos\varphi}{i_1} & -\dfrac{\sin\varphi}{i_2} & 0 \\ -\dfrac{\sin\varphi\cos\theta}{i_1 \sin\theta} & -\dfrac{\cos\varphi\cos\theta}{i_2 \sin\theta} & \dfrac{1}{i_3^2} \end{pmatrix}.
\end{aligned}
$$

From (28), by a simple calculus, we find

$$
\begin{cases}
w_{23}^1 = \dfrac{i_1}{i_2 i_3}\ ,\ w_{31}^2 = \dfrac{i_2}{i_1 i_3}\ ,\ w_{12}^3 = \dfrac{i_3}{i_1 i_2}\ , \\
w_{32}^1 = -\dfrac{i_1}{i_2 i_3}\ ,\ w_{13}^2 = -\dfrac{i_2}{i_1 i_3}\ ,\ w_{21}^3 = -\dfrac{i_3}{i_1 i_2}\ ,
\end{cases}
$$

which substituted in (27) give us the Levi-Civitas connection coefficients already calculated in the equations (26).

We ascertain that the Riccis rotation coefficients coincide with the Levi–Civitas connection coefficients (26).

The derivatives of the orthogonal basis $\vec{e}_1'$, $\vec{e}_2'$ in the vector field $\dfrac{dx^\mu}{ds} = i_\mu \omega^\mu$; $\mu = 1, 2, 3$ are

$$
\frac{d\vec{e}_h'}{ds} = \gamma_{h\mu}^\lambda \frac{dx^\mu}{ds} \vec{e}_\lambda'
$$

respectively

$$(30)\qquad \begin{cases} \dfrac{d\vec{e}_1'}{ds} = \gamma_{1\mu}^{\lambda}\dfrac{dx^{\mu}}{ds}\vec{e}_{\lambda}' = \gamma_{13}^{2}\dfrac{dx^{3}}{ds}\vec{e}_2' + \gamma_{12}^{3}\dfrac{dx^{2}}{ds}\vec{e}_3' = \\ \qquad = \dfrac{i_1^2+i_2^2-i_3^2}{2i_1i_2}\omega_3\vec{e}_2' + \dfrac{i_2^2-i_3^2-i_1^2}{2i_1i_3}\omega_3\vec{e}_3' \\ \dfrac{d\vec{e}_2'}{ds} = \gamma_{2\mu}^{\lambda}\dfrac{dx^{\mu}}{ds}\vec{e}_{\lambda}' = \gamma_{23}^{1}\dfrac{dx^{3}}{ds}\vec{e}_1' + \gamma_{21}^{3}\dfrac{dx^{1}}{ds}\vec{e}_3' = \\ \qquad = \dfrac{i_3^2-i_1^2-i_2^2}{2i_1i_2}\omega_3\vec{e}_1' + \dfrac{i_2^2+i_3^2-i_1^2}{2i_2i_3}\omega_1\vec{e}_3' \\ \dfrac{d\vec{e}_3'}{ds} = \gamma_{3\mu}^{\lambda}\dfrac{dx^{\mu}}{ds}\vec{e}_{\lambda}' = \gamma_{32}^{1}\dfrac{dx^{2}}{ds}\vec{e}_1' + \gamma_{31}^{3}\dfrac{dx^{1}}{ds}\vec{e}_2' = \\ \qquad = \dfrac{i_3^2+i_1^2-i_2^2}{2i_1i_2}\omega_2\vec{e}_1' + \dfrac{i_1^2-i_2^2-i_3^2}{2i_2i_3}\omega_1\vec{e}_2' \ . \end{cases}$$

These equations can be written in a matricial form

$$(31)\qquad \begin{pmatrix} \dfrac{d\vec{e}_1'}{ds} \\ \dfrac{d\vec{e}_2'}{ds} \\ \dfrac{d\vec{e}_3'}{ds} \end{pmatrix} = \gamma \begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{pmatrix}$$

where γ denotes the skew-symmetric matrix

$$(32)\quad \gamma = \begin{pmatrix} 0 & \dfrac{i_1^2+i_2^2-i_3^2}{2i_1i_2}\omega_3 & \dfrac{i_2^2-i_3^2-i_1^2}{2i_1i_3}\omega_2 \\ \dfrac{i_3^2-i_1^2-i_2^2}{2i_1i_2}\omega_3 & 0 & \dfrac{i_2^2+i_3^2-i_1^2}{2i_1i_2}\omega_1 \\ \dfrac{i_3^2+i_1^2-i_2^2}{2i_1i_3}\omega_2 & \dfrac{i_1^2+i_2^2-i_3^2}{2i_2i_3}\omega_1 & 0\ . \end{pmatrix}$$

If we also introduce the vector $\vec{\gamma} = \gamma_1\vec{e}_1' + \gamma_2\vec{e}_2' + \gamma_3\vec{e}_3'$ with

$$\gamma_3 = \frac{1}{2}(\gamma_1^2 - \gamma_2^1) \quad , \quad \gamma_2 = \frac{1}{2}(\gamma_3^1 - \gamma_1^3) \quad , \quad \gamma_1 = \frac{1}{2}(\gamma_2^3 - \gamma_3^2)$$

we have

$$\frac{d\vec{e}'_\alpha}{ds} = \gamma \times \vec{e}'_\alpha \ . \tag{33}$$

2 Absolute frame. Frame of "observation"

If we make a precis of the first paragraph, we have constructed the metric (9) of V_3 starting from the kinetic energy of rotation.

To calculate this kinetic energy of rotation we have considered the existence of an orthogonal frame of reference (absolute) and we have expressed the energy as function on the Eulers angles between the central axes and the axes of the absolute frame.

In the structure of V_3 we have defined a natural basis (12) and we made the change of basis (18) such that the components of the angular moments with respect to the basis (18) coincide with the components of the angular moment with respect to the central axes.

Finally, in (24), we have normalized the basis (18) obtaining $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$. If we write the equations of the geodesics (14) in the basis $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ we find again the Eulers equations. Indeed, we have

$$(34)\begin{cases} \dfrac{d}{dt}i_1\omega_1' + \dfrac{i_3^2 - i_1^2 - i_2^2}{2i_1i_2i_3}i_2\omega_2'i_3\omega_3' + \dfrac{i_3^2 + i_1^2 - i_2^2}{2i_1i_2i_3}i_3\omega_3'i_2\omega_2' = 0 \\ \dfrac{d}{dt}i_2\omega_2' + \dfrac{i_1^2 - i_2^2 - i_3^2}{2i_1i_2i_3}i_3\omega_3'i_1\omega_1' + \dfrac{i_1^2 + i_2^2 - i_3^2}{2i_1i_2i_3}i_1\omega_1'i_3\omega_3' = 0 \\ \dfrac{d}{dt}i_3\omega_3' + \dfrac{i_2^2 - i_3^2 - i_1^2}{2i_1i_2i_3}i_1\omega_1'i_2\omega_2' + \dfrac{i_2^2 + i_3^2 - i_1^2}{2i_1i_2i_3}i_2\omega_2'i_1\omega_1' = 0 \end{cases}$$

and, after making the calculus, we obtain

$$(35)\qquad \begin{cases} \dfrac{d}{dt}i_1^2\omega_1' + (i_3^2 - i_3^2)\omega_2'\omega_3' = 0 \\ \dfrac{d}{dt}i_2^2\omega_2' + (i_1^2 - i_3^2)\omega_1'\omega_3' = 0 \\ \dfrac{d}{dt}i_3^2\omega_3' + (i_2^2 - i_1^2)\omega_1'\omega_2' = 0 \end{cases}$$

which represent the Eulers equations.

Let identify now the orthogonal basis $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ with the directions of the central axes $\vec{i}', \vec{j}', \vec{k}'$ and let determine the rotated frame of matrix (2) with respect to $\vec{e}_1', \vec{e}_2', \vec{e}_3'$. We denote the unit vectors of this frame by $\vec{e}_1', \vec{e}_2', \vec{e}_3'$. In this hypothesis $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ ought to represent the unit vectors of the absolute frame.

From the equation

$$(36)\qquad \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} = A \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3, \end{pmatrix}$$

taking the derivative we have

$$\begin{bmatrix} \dfrac{d\vec{e}_1}{ds} \\ \dfrac{d\vec{e}_2}{ds} \\ \dfrac{d\vec{e}_3}{ds} \end{bmatrix} = \frac{dA}{ds} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} + A \begin{bmatrix} \dfrac{d\vec{e}_1}{ds} \\ \dfrac{d\vec{e}_2}{ds} \\ \dfrac{d\vec{e}_3}{ds} \end{bmatrix}$$

In view of (31) and (36), the preceding equation can be written in the form

$$\begin{bmatrix} \dfrac{d\vec{e}_1}{ds} \\ \dfrac{d\vec{e}_2}{ds} \\ \dfrac{d\vec{e}_3}{ds} \end{bmatrix} = \frac{dA}{ds} A^{-1} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} + A\gamma A^{-1} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}$$

respectively

$$\begin{bmatrix} \dfrac{d\vec{e}_1}{ds} \\ \dfrac{d\vec{e}_2}{ds} \\ \dfrac{d\vec{e}_3}{ds} \end{bmatrix} = A \left(A^{-1} \frac{dA}{ds} + \gamma \right) A^{-1} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \tag{37}$$

As the components of the skew-symmetric tensor ω from (4)

represent the product $\omega = \dfrac{dA^{-1}}{ds}A$, it follows that

$$A^{-1}\frac{dA}{ds} = -\omega = \begin{pmatrix} 0 & -\omega_3' & \omega_2' \\ \omega_3' & 0 & -\omega_1' \\ -\omega_2' & \omega_1' & 0 \end{pmatrix} \tag{38}$$

Let denote $\Omega' = A^{-1}\dfrac{dA}{ds} + \gamma$.

In view of (32) and (38) we obtain

$$\Omega' = \tag{39}$$

$$= \begin{pmatrix} 0 & \dfrac{(i_1-i_2)^2-i_3^2}{2i_1i_2i_3}i_3\omega_3' & -\dfrac{(i_3-i_1)^2-i_2^2}{2i_1i_2i_3}i_2\omega_2' \\ -\dfrac{(i_1-i_2)^2-i_3^2}{2i_1i_2i_3}i_3\omega_3' & 0 & \dfrac{(i_2-i_3)^2-i_1^2}{2i_1i_2i_3}i_1\omega_1' \\ \dfrac{(i_3-i_1)^2-i_2^2}{2i_1i_2i_3}i_2\omega_2' & \dfrac{(i_2-i_3)^2-i_1^2}{2i_1i_2i_3}i_1\omega_1' & 0 \end{pmatrix}$$

The above matrix is a skew-symmetric one and we will denote by

$$\begin{cases} \Omega_1' = \Gamma_3'^2 = \dfrac{(i_2-i_3)^2-i_1^2}{2i_1i_2i_3}i_1\omega_1' \\ \Omega_2' = \Gamma_1'^3 = \dfrac{(i_3-i_1)^2-i_2^2}{2i_1i_2i_3}i_2\omega_2' \\ \Omega_3' = \Gamma_2'^2 = \dfrac{(i_1-i_2)^2-i_3^2}{2i_1i_2i_3}i_3\omega_3' \end{cases} \tag{40}$$

If we calculate $A\left(A^{-1}\dfrac{dA}{ds} + \gamma\right)A^{-1} = A\Omega' A^{-1}$ we obtain also

a skew–symmetric matrix Ω,

$$\Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma_1^1 & \Gamma_1^2 & \Gamma_1^3 \\ \Gamma_2^1 & \Gamma_2^2 & \Gamma_2^3 \\ \Gamma_3^1 & \Gamma_3^2 & \Gamma_3^3 \end{pmatrix} \tag{41}$$

and we have

$$\begin{cases} \Omega_1 = \Gamma_2^3 = a_1^1\Omega_1' + a_1^2\Omega_2' + a_1^3\Omega_3' \\ \Omega_2 = \Gamma_3^1 = a_2^1\Omega_1' + a_2^2\Omega_2' + a_2^3\Omega_3' \\ \Omega_3 = \Gamma_1^2 = a_3^1\Omega_1' + a_3^2\Omega_2' + a_3^3\Omega_3' \end{cases} \tag{42}$$

where $\lfloor a_j^i \rfloor = A(\psi, \theta, \varphi)$ given in (2).

With the above notations, the equation (37) becomes

$$\begin{bmatrix} \dfrac{d\vec{e}_1}{ds} \\ \dfrac{d\vec{e}_2}{ds} \\ \dfrac{d\vec{e}_3}{ds} \end{bmatrix} = \Omega \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \tag{43}$$

respectively

$$\frac{d\vec{e}_\alpha}{ds} = \Gamma_\alpha^\beta \vec{e}\beta \ . \tag{44}$$

If we introduce the vector $\vec{\Omega} = \Omega_1\vec{e}_1 + \Omega_2\vec{e}_2 + \Omega_3\vec{e}_3$, for $\dfrac{d\vec{e}_\alpha}{ds}$ we obtain the equation of Poisson

$$\frac{d\vec{e}_\alpha}{ds} = \vec{\Omega} \times \vec{e}\alpha \ . \tag{45}$$

We remark that, after we have identified the orthogonal basis $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ with the directions of the central axes, the rotation of matrix A given by (2) of this basis ought to leads us to the absolute frame $(\vec{i}, \vec{j}, \vec{k})$.

As the derivatives of $\vec{e}_1, \vec{e}_2, \vec{e}_3$ satisfy (45), these vectors are not fixed vectors and we will call the frame associated to $\vec{e}_1, \vec{e}_2, \vec{e}_3$, frame of "observation".

Once again, we underline that in view of (43), (45) *it is not an absolute frame.*

On the other hand, from (36), we have

$$\begin{bmatrix} \vec{e}_1'^{pr} \\ \vec{e}_2'^{pr} \\ \vec{e}_3'^{pr} \end{bmatrix} = A^{-1} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \tag{46}$$

and, if we take into consideration the rotation of $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ with respect to the frame of "observation" $\vec{e}_1, \vec{e}_2, \vec{e}_3$, we have ($\frac{d'}{ds}$ denotes the derivative with respect to the frame of "observation")

$$\begin{bmatrix} \frac{d\vec{e}_1'}{ds} \\ \frac{d\vec{e}_2'}{ds} \\ \frac{d\vec{e}_3'}{ds} \end{bmatrix} = \frac{dA^{-1}}{ds} A \begin{bmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{bmatrix} = \omega \begin{bmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{bmatrix} \tag{47}$$

where ω is the matrix given by (4).

In the equations (46), (47) the components of ω given by (5), have the significance of the components of the angular velocity of rotation of the frame $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ (assimilated to the central axes) with respect to the frame of " observation".

The angular velocity of rotation of the frame of "observation" given by (40) can also be written

$$(48) \qquad \begin{cases} \Omega_1' = \dfrac{(i_2 - i_3 - i_1)(i_2 - i_3 + i_1)}{2i_2 i_3}\omega_1' \\ \Omega_2' = \dfrac{(i_3 - i_1 - i_2)(i_3 - i_1 + i_2)}{2i_1 i_3}\omega_2' \\ \Omega_3' = \dfrac{(i_1 - i_2 - i_3)(i_1 - i_2 + i_3)}{2i_1 i_2}\omega_3' \end{cases}$$

If we suppose $i_1^2 < i_2^2 < i_3^2$ and in the above equations we take into consideration the inequalities

$$(49) \qquad \begin{aligned} i_1^2 < i_2^2 + i_3^2 < (i_2 + i_3)^2 \ , \\ i_2^2 < i_1^2 + i_3^2 < (i_1 + i_3)^2 \ , \\ i_3^2 < i_1^2 + i_2^2 < (i_1 + i_2)^2 \end{aligned}$$

which imply

$$(50) \qquad i_1 < i_2 + i_3 \quad , \quad i_2 < i_1 + i_3 \quad , \quad i_3 < i_1 + i_2$$

(i.e. the coefficients of $\omega_1', \omega_2', \omega_3'$ are negative) it follows that the angular velocity of the frame of "observation" has an opposite sense with respect to the angular velocity $\omega_1', \omega_2', \omega_3'$.

The well known solutions of the Eulers equations are, [2],

$$\begin{aligned} \omega_3' &= -\gamma\, cnm_1(s - s_0)\ , \\ \omega_2' &= \alpha\, snm_1(s - s_0)\ , \\ \omega_1' &= \delta\, dnm_1(s - s_0) \end{aligned} \tag{51}$$

where s_0 is the value of s for which $\omega_2' = 0$,

$$m_1 = K\sqrt{\frac{d(i_3 - d)(i_2 - i_1)}{i_1 i_2 i_3}},\quad \alpha = K\sqrt{\frac{d(d - i_1)}{i_2(i_2 - i_1)}},$$

$$\gamma = K\sqrt{\frac{d(d - i_1)}{i_3(i_3 - i_1)}},\qquad \delta = K\sqrt{\frac{d(i_3 - d)}{i_1(i_3 - i_1)}},$$

and K, d are constants of integration, $dK = K_c^0$ with $i_1 < d < i_2$.

The three elliptical functions from (51) vary periodically with the period $4T\,/m_1$ for ω_2' and ω_3' respectively $2T\,/m_1$ for ω_1' where

$$T = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - kx^2)}}$$

and

$$k = \frac{(d - i_1)(i_3 - i_2)}{(i_3 - i_1)(i_3 - d)}\ .$$

The solutions (51), expressed through the intermedium of elliptical functions, represent the components of the angular velocity with respect to the frame of "observation".

As a very simplified physical interpretation of the above result we could consider the following example.

A material point executes a rectilinear translation with the absolute velocity $\vec{\nu} = \nu_0(1-e^{-1})\vec{i}$ and a frame of "observation" makes a translation in the opposite sense with $\vec{\nu}_{obs} = -\nu_0 e^{-1}\vec{i}$ with respect to the absolute frame.

In these circumstances, with respect to the frame of "observation", the material point has an uniform translation with $\vec{\nu}_{rel} = -\nu_0\vec{i}$.

In what sense the above example is connected to the periodical solutions (51) of the Eulers equations?

In the structure of the variety V_3 they are solutions with respect to the frame of "observation" which revolves in the opposite sense with respect to the rotation of the central axes so that the angular velocity, which varies periodically with respect to the frame of "observation", could decrease with respect to an absolute frame.

The result given by (49), (51) leads us to the conclusion that the frame associated to the central axes has an angular velocity which varies periodically following (51), but it is possible only taking into account that the frame of "observation" revolves in opposite sense, to make possible this fact.

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