General Mathematics Vol. 8, No. 1–2 (2000)

Note on some properties of Lusternik–Schnirelmann category

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Abstract

Some basic properties concerning the Lusternik - Schnirelmann category of a topological sum, respectively of a quotient space are given in [1] and [3] .

1991 Mathematical Subject Classification 58E05

1 Introduction

Let X be a topological space and let $A \subseteq X$ be a closed subset.

Definition: *A has category* m $(m \in Z)$ *in* X *if* $A = \bigcup_{k=1}^{m} A_k$, *each subset* A_k *is closed contractible in* X *and* m *is the minimal integer with this property. In this case* m *is denoted by cat* (A, X). *If* $A = X$, *then cat* (X, X)*is shortly denoted by cat* (X) *and it is called the Lusternik– Schnirelmann category of space* X *(L-S category in short). In case when the integer mis not finite, we put cat* $(X) = \infty$.

There is a rich literature concerning L-S category and its generalizations (see for instance the paper [1]).

Some important examples of spaces for which the L-S category can be effectively determined are given by:

$$cat\,(D^n) = 1, \qquad cat\left(S^{n-1}\right) = 2,$$

$$cat\,(T^n) = n+1, \qquad cat\,(P^n\,(R)) = n+1,$$

where $D^n = \left\{x \in R^{n+1} : \|x\| \leq 1\right\}$ (the n-dimensional closed ball), $S^n = \left\{x \in R^{n+1} : \|x\| = 1\right\} = \partial D^n$ (the n-dimensional sphere), $T^n = \underbrace{S^1 \times ... \times S^1}_{n \text{ times}}$ (the n–dimensional torus), $P^n\,(R)$- the n–dimensional projective space. The determination of L-S category for certain classes of topological spaces is still an important research problem.

One of the famous applications of L-S category is given by the so called L-S multiplicity critical points theorem (see [2]).

Theorem: *Let M be a Hilbert manifold and let $f : M \to R$ be a differentiable mapping satisfying the following conditions (which are called compactness conditions):*

(a) M is a Riemannian complete manifold;

(b) $M\inf f > -\infty$ (i.e. f is bounded from bellow);

(c) (Palais–Smale condition (C)) Every sequence $(x_n)_{n\geq 0}$ *in* M *such that* $(f(x_n))_{n\geq 0}$ *is bounded and* $\left\|(df)_{x_n}\right\| \to 0$*, has a convergent subsequence* $\left(x_{n_k}\right)_{k\geq 0}$ *in* M*.*

Then the critical set of f*,*

$$C(f) = \{x \in M : (df)_x = 0\}$$

contains at least $cat\,(M)$ *critical points.*

2 Main results

The following general properties of the above mentioned notion of category are well-known (see [2] for proofs and details):

1. If $A \subseteq B \subseteq X$ are closed sets, then

$$cat\,(A, X) \leq cat\,(B, X) \quad \text{(the monotony)}$$

2. If $A, B \subseteq X$ are closed sets, then

$$cat\,(A \cup B, X) \leq cat\,(A, X) + cat\,(B, X)$$

(the subadditivity property)

3. If $A \subseteq B \subseteq X$ are closed sets, then

$$cat\left(\overline{B \backslash A}, X\right) \geq cat\,(B, X) - cat\,(A, X)$$

4. If $A \subset B \subseteq X$ are closed, B being deformable onto A, then
$$cat\,(B, X) \leq cat\,(A, X)$$

5. If $A \subseteq X$ is closed and $h : X \to X$ is a homeomorphism, then
$$cat\,(h\,(A)\,, X) = cat\,(A, X)$$

6. For two topological spaces X_1, X_2 and $A_1 \subseteq X_1, A_2 \subseteq X_2$ closed subsets, the inequality
$$cat\,(A_1 \times A_2, X_1 \times X_2) \leq cat\,(A_1, X_1) \cdot cat\,(A_2, X_2)$$
holds.

In what follows we obtain some supplementary properties in this meaning. Recall that the topological sum of a family $\{X_i\}_{i \in I}$ of spaces is the space defined by
$$\bigsqcup_{i \in I} X_i = \{(i, x_i) : x_i \in X_i, i \in I\}$$
(see [4] for more details).

Proposition 1 : *For the topological spaces X_1, X_2 the relation*

$$cat\left(X_1 \bigsqcup X_2\right) \leq cat\,(X_1) + cat\,(X_2) \tag{1}$$

is true.

Proof By using definition of the topological sum one obtains

$$X_1 \bigsqcup X_2 = \{(1, x_1), (2, x_2) : x_1 \in X_1, x_2 \in X_2\} =$$

$$= \{(1, x_1) : x_1 \in X_1\} \cup \{(2, x_2) : x_2 \in X_2\} =$$

$$= (\{1\} \times X_1) \cup (\{2\} \times X_2)$$

Consider $m = cat(X_1), n = cat(X_2)$. Then there exist the closed contractible subsets

$$A_k \subset X_1, B_l \subset X_2, k = \overline{1, m}, l = \overline{1, n},$$

such that

$$X_1 = \bigcup_{k=1}^{m} A_k, X_2 = \bigcup_{l=1}^{n} B_l.$$

It follows

$$X_1 \bigsqcup X_2 = (\{1\} \times X_1) \cup (\{2\} \times X_2) =$$

$$= \left(\{1\} \times \bigcup_{k=1}^{m} A_k\right) \cup \left(\{2\} \times \bigcup_{l=1}^{n} B_l\right) =$$

$$= \left(\bigcup_{k=1}^{m} (\{1\} \times A_k)\right) \cup \left(\bigcup_{l=1}^{n} (\{2\} \times B_l)\right)$$

It is clear that the sets $\{1\} \times A_k, \{2\} \times B_l, k = \overline{1, m}, l = \overline{1, n}$ are closed in $X_1 \bigsqcup X_2$. Let us prove that these sets are contractible, too in this space. Because A_k is contractible in X_1 we have $i \simeq c_{x_k}$, where $i : A_k \hookrightarrow X_1$ is the inclusion map and c_{x_k} is the constant map. Define

$$1 \times i : \{1\} \times A_k \to X_1 \bigsqcup X_2$$

$$(1 \times i)(1, a) = (1, i(a)), \forall a \in A_k, k = \overline{1, m}$$

and obtain $1 \times i \simeq 1 \times c_{x_k}, k = \overline{1, m}$. In an analogous way it follows that $\{2\} \times B_l$ are contractible in $X_1 \bigsqcup X_2$.

Therefore $X_1 \bigsqcup X_2$ is the reunion of $m + n$ closed contractible sets in $X_1 \bigsqcup X_2$, i.e. the inequality (1) is proved.

Remark The inequality (1) can be extended for a finite family of topological space $\{X_i\}_{i \in I}$ as follows:

$$cat\left(\bigsqcup_{i \in I} X_i\right) \leq \sum_{i \in I} cat(X_i) \tag{2}$$

If X is a topological space and $A \subseteq X$, we define on X the following equivalence relation:

$$x_1 \rho x_2 \Leftrightarrow x_1, x_2 \in A \text{ or } x_1 = x_2.$$

Definition: *The quotient space X/ρ is marked with X/A and it is called the space obtained from X by identification of A to a point* (see [4]).

Proposition 2 : *The following inequality holds:*

$$cat(X) - cat\left(\overline{A}, X\right) \leq cat(X/A). \tag{3}$$

Proof By using the definition of the quotient space one obtains

$$X/A = \{\hat{x} : x \in X\},$$

where

$$\hat{x} = \{y \in X : y\rho x\} = \{y \in X : x, y \in A \text{ or } y = x\} \ ;$$

$$X/A = \{A, \{x\} : x \in X \backslash A\}.$$

We have

$$X \backslash A \subseteq X/A \quad \text{and} \quad cat\left(\overline{X \backslash A}\right) \leq cat\,(X/A)\,.$$

Using the property 1) and 3), we obtain:

$$cat(X/A) \geq cat(\overline{X \backslash A}, X) \geq cat(X) - cat(A, X).$$

Application: We consider that:

$$X = S^n \times I, I = [0,1], A = S^n \times \{0\}.$$

We have:

$$cat(S^n \times I) - cat(S^n \times \{0\}) \leq cat(S^n \times I/S^n \times \{0\})$$

and

$$cat(S^n \times \{0\}) = 2.$$

We define:

$$\pi : S^n \times I \longrightarrow S^n \times I/S^n \times \{0\}\ ,\ f : S^n \times I \longrightarrow D^{n+1},$$

where $f(t,x) = tx$. There is a homeomorphism

$$g : S^n \times I/S^n \times \{0\} \longrightarrow D^{n+1}$$

such that $g \circ f = \pi$. Then $S^n \times I/S^n \times \{0\}$and D^{n+1} are homeomorphic and we obtain

$$cat\,(S^n \times I/S^n \times \{0\}) = cat\left(D^{n+1}\right) = 1.$$

The relation (3) becomes

$$cat\,(S^n \times I) - 2 \leq 1$$

that is

$$cat\,(S^n \times I) \leq 3$$

On the other hand $S^n \times I$ is simple connected: $\pi_1(S^n \times I) = 0$ and we have

$$\pi_n(S^n \times I) \cong \pi_n(S^n) \times \pi_n(I)\,; \pi_n(S^n) \cong Z$$

and

$$\pi_n(S^n \times I) \cong Z.$$

We conclude that $S^n \times I$ is not contractible. Then $cat(S^n \times I) > 1$. So we have

$$2 \leq cat(S^n \times I) \leq 3 \tag{4}$$

The next problem is interesting: *which number is* $cat(S^n \times I)$*?* The following conjecture of W. Singhof is known as being

$$cat\,(X \times S^n) = cat\,(X) + 1$$

(see [5]) .

If the conjecture is true, then

$$cat\,(S^n \times I) = cat\,(I) + 1 = 2$$

Also, the following open problems are interesting:

1. The finding of a relation for the category of an adjunction space $B_f \bigsqcup A$ in terms of categories of the spaces A and B (see [4]).

2. The similar problem can be formulated for a connected sum $X \# Y$ of topological spaces (see[4]) .

References

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