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On a class of convex functions in a half-plane

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Abstract

In this paper we obtain some properties of a class of convex univalent functions in a half-plane.

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1 Introduction

Let D be the half-plane $\{z : z \in \mathbb{C}, Im\,(z) > 0\}$ and let $\mathcal{H}$ be the class of analytic functions in the domain D.

We denote by $\mathcal{H}_1$ the familly of the functions f of the class $\mathcal{H}$, which are normalized by the condition

$$(1) \qquad \lim_{D \ni z \to \infty} [f\,(z) - z] = 0$$

The above normalization (1) is related to fluid mechanics and is named hydrodinamically normalization.

J.Stankiewicz and Z.Stankiewicz obtained sufficient conditions for analytic functions of the class $\mathcal{H}_1$ to be convex and univalent in the domain D :

Theorem 1 ([3], [4]) *If the function f belongs to the class $\mathcal{H}_1$, $f'(z) \neq 0$ and*

$$Im\frac{f''(z)}{f'(z)} > 0 \tag{2}$$

for all $z \in D$, then the function f is convex and univalent in the domain D.

An other class of convex univalent functions was defined in [2], considering analytic functions in the domain D, not hydrodinamically normalized:

Theorem 2 ([2]) *If the function f belongs to the class $\mathcal{H}$, $f'(z) \neq 0$*

$$Re\frac{zf''(z)}{f'(z)} + 1 > 0 \tag{3}$$

and

$$Im\frac{f''(z)}{f'(z)} > 0 \tag{4}$$

for all $z \in D$, then the function f is convex and univalent in D.

It is well-known that, if g is a convex univalent function in the unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$, then

$$(5) \qquad \left| -2\left|z\right|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \leq 2$$

for all $z \in U$.

A similar result can be obtained for the class C_1 of convex univalent functions in the domain D, defined as follows:

Definition 1 *We say that the function $f \in \mathcal{H}$ is in the class C_1, if the function f satisfies the inequalities* (3), (4) *and* $f'(z) \neq 0$ *for all* $z \in D$.

For this is necessary the folowing lemma:

Lema 1 *Let D_r be the family of increased domains*

$$D_r = \{z : z \in \mathbb{C}, |z - z_r| < R_r\}$$

where

$$z_r = i\frac{1+r^2}{1-r^2}, R_r = \frac{2r}{1-r^2}, 0 < r < 1$$

For all fixed point $z \in D$ there exists $r_z \in (0,1)$, such that $z = z_r + R_r u_r \in D_r$ for all $r > r_z$ and

$$\lim_{r\to 1}\left(1 - |u_r|\right)R_r = Im\left(z\right), \lim_{r\to 1} u_r = -i$$

2 Main results

Let $g : U \to D$ be the function defined by

$$g(u) = i\frac{1-u}{1+u}$$

and let $U_r = \{u : u \in \mathbb{C}, |u| < r\}$, where $0 < r < 1$ and $U_1 = U$.

For all fixed $r \in (0,1)$ and $u = re^{it}$, $t \in [0, 2\pi]$ we have

$$\left| i\frac{1-u}{1+u} - i\frac{1+r^2}{1-r^2} \right| = \left| \frac{2r\left(r + e^{it}\right)}{\left(1 + re^{it}\right)\left(1 - r^2\right)} \right| = \frac{2r}{1-r^2}$$

because

$$\left| r + e^{it} \right| = \left| re^{-it} + 1 \right| = \left| re^{it} + 1 \right|$$

It results that for all r, $0 < r < 1$, $g(U_r) = D_r$.We observe that $g(U) = D$.

Theorem 3 *If the function f belongs to the class C_1 then for all $r \in (0,1)$, $f(D_r)$ is a convex domain.*

Proof. If the function f belongs to the class C_1, from Theorem 2 we have that f is a convex univalent function in the domain D. Because $f(D)$ is a convex domain and $D = g(U,)$ it results that $(f \circ g)(U) = f(D)$ is a convex domain.

The function $h : U \to \mathbb{C}$ defined from

$$h(u) = (f \circ g)(u) - f(i)$$

is a convex univalent function in U and hence, for all $r \in (0,1)$, $h(U_r) = f(g(U_r)) - f(i) = f(D_r) - f(i)$ is a convex domain.We conclude that $f(D_r)$ is a convex domain.

Theorem 4 *If the function f belongs to the class C_1, then*

$$\left| i - Im(z)\frac{f''(z)}{f'(z)} \right| \leq 1 \tag{6}$$

for all $z \in D$.

Proof. Let $z \in D$ be a fixed point.From Lemma it results that there exists $r_z \in (0,1)$ such that $z = z_r + R_r u_z$, for all $r > r_z$.

We consider the familly of functions $g_r : U \to \mathbb{C}$, $t \in I$, defined from

$$g_r(u) = f(z_r + R_r u) - f(z_r)$$

Because for all fixed $r \in (0,1)$ the domain $g_r(U)$ is obtained from f(D_r) by a translation given by $-f(z_r)$ and $f(D_r)$ is a convex domain, it results that the domain $g_r(U)$ is a convex domain.The functions f and g are univalent functions and hence, for all fixed $r \in (0,1)$ the function g_r is an univalent function in U.

For $r > r_z$, from inequality (5) we obtain

$$\left| -2|u_r|^2 + \left(1 - |u_r|^2\right)\frac{u_r g_r''(u_r)}{g_r'(u_r)} \right| =$$

$$= \left| -2 \left| u_r \right|^2 + \left(1 - \left| u_r \right|^2 \right) R_r \frac{u_r f''(z)}{f'(z)} \right| \leq 2.$$

From Lemma, for $r \to 1$ we obtain

$$\left| -2 - 2iIm(z) \frac{f''(z)}{f'(z)} \right| \leq 2$$

and the theorem is proved.

Remark 1 *The function $f : D \to \mathbb{C}$, defined by $f(z) = z$ belongs to the class C_1. For this functions and for $z \in D$ we have*

$$\left| i - Im(z) \frac{f''(z)}{f'(z)} \right| = 1$$

It results that the inequlity (6) *is sharp.*

Corollary 1 *If the function f belongs to the class C_1, then for all fixed $x \in \mathbb{R}$, the function $m : I \to \mathbb{R}, I = [0, \infty)$ defined by*

$$m(y) = |f'(x + iy)|$$

is a decreasing function and the function $M : I \to \mathbb{R}$

$$M(y) = y^2 |f'(x + iy)|$$

is an increasing function on I.

Proof. From Theorem2.2 it follows that, for all $x \in \mathbb{R}$ and all $y > 0$ we have

$$\left| 1 + y \frac{\partial}{\partial y} log f'(x + iy) \right| \leq 1$$

and hence

$$-\frac{2}{y} \leq \frac{\partial}{\partial y} ln \left| f' \left(x + iy \right) \right| \leq 0$$

It results that

$$\frac{\partial}{\partial y} ln \left| y^2 f' \left(x + iy \right) \right| \geq 0$$

and hence

$$m' \left(y \right) \leq 0, M' \left(y \right) \geq 0$$

The corollary is proved.

Corollary 2 *If the function f belongs to the class C_1, then for all fixed $y > 0$ the function $n : \mathbb{R} \to \mathbb{R}$ defined by*

$$n \left(x \right) = -\frac{x}{y} + ln \left| f' \left(x + iy \right) \right|$$

is a decreasing function and the function $N : \mathbb{R} \to \mathbb{R}$ defined by

$$N \left(x \right) = \frac{x}{y} + ln \left| f' \left(x + iy \right) \right|$$

is a increasing function.

Proof. The inequality (6) can be written as

$$\left| i - y\frac{\partial}{\partial x} log f'(x+iy) \right| \leq 1$$

and hence, for all fixed $y > 0$.

It results that

$$-\frac{1}{y} \leq \frac{\partial}{\partial x} ln\left|f'(x+iy)\right| \leq \frac{1}{y}$$

and hence

$$n'(x) \leq 0, N'(x) \geq 0$$

From the above inequalityes it results that n is a decreasing function and N is an increasing function.

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