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On an integral operator and starlike functions of order $1/2$

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Abstract

We consider the following integral operator:

$$F(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z f(t) t^{\gamma-1} dt, \quad \gamma \geq 0. \tag{1}$$

In this paper we will obtain some conditions on f such that F be a starlike functions of order $1/2$.

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1 Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbf{C},\ |z| < 1\},$$

let $\mathcal{H}[U]$ be the space of holomorphic functions in U. We let:

$$A = \{f \in \mathcal{H}[U],\ f(z) = z + a_2 z^2 + a_{n+2} z^{n+2} + \ldots,\ z \in U\}$$

and

$$S^* = \left\{ f \in A,\ \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$

the class of starlike functions in U

$$S^*(\alpha)=\left\{f\in A,\ \operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha,\ z\in U\right\}$$

the class of starlike functions of order α, $0\le\alpha<1$.

If f and g are analytic functions in U, then we say that f is subordinated to g, written $f\prec g$ or $f(z)\prec g(z)$, if there is a function w analytic in U with $w(0)=0$, $|w(z)|<1$ for all $z\in U$ such that $f(z)=g(w(z))$ for $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Lemma A. [1,2] *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$, when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi[q(z)],$$

$$h(z)=\theta[q(z)]+Q(z)$$

and suppose that

(i) Q is starlike

and

(ii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)}=\operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]}+\dfrac{zQ'(z)}{Q(z)}\right]>0.$

If p is analytic in U, with $p(0)=q(0)$, $p(U)\subset D$ and

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec\theta[q(z)]+zq'(z)\phi[q(z)]=h(z),$$

then $p\prec q$, and q is the best dominant.

2 Main results

Theorem. *Let* $\gamma \geq 0$ *and let*

(2) $$h(z) = \frac{1}{1-z} + \frac{z}{(1-z)(1+\gamma-\gamma z)}, \quad z \in U.$$

If $f \in A$ *and*

(3) $$\frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > \frac{1}{2},$$

where F *is given by (1).*

Proof. From (1) we deduce

(4) $$\gamma F(z) + zF'(z) = (\gamma+1) f(z).$$

If we let $p(z) = \dfrac{zF'(z)}{F(z)}$, then (4) becomes

$$\gamma F(z) + p(z)F(z) = (\gamma+1) f(z),$$

and we obtain

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}.$$

Then (3) becomes

(5) $$\frac{zp'(z)}{p(z)} + p(z) \prec h(z).$$

To show that $F \in S^*(1/2)$, we will use Lemma A.

If we let

$$q(z) = \frac{1}{1-z},$$

then

$$\phi[q(z)] = \frac{1}{q(z)+\gamma} = \frac{1-z}{1+\gamma-\gamma z}$$
$$\theta[q(z)] = q(z) = \frac{1}{1-z}$$
$$Q(z) = zq'(z)\phi[q(z)] = \frac{z}{(1-z)(1+\gamma-\gamma z)}$$
$$h(z) = \theta[q(z)] + Q(z) = \frac{1}{1-z} + \frac{z}{(1-z)(1+\gamma-\gamma z)}.$$

Since Q is starlike and Re $\phi(q(z)) > 0$, by Lemma A we deduce $p \prec q$ which is equivalent to

$$\mathrm{Re}\,\frac{zF'(z)}{F(z)} > \frac{1}{2}\,.$$

References

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