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Some convergence properties in the Hausdorff-Pompeiu metric

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Abstract

The space $\mathcal{K}(X)$ of all non-empty compact subsets of a complete metric space endowed with the Hausdorff-Pompeiu metric δ have been used extensively during the last century in the fractal theory.

In this paper we will present some results relative to the convergence of a sequence in the metric space $(\mathcal{K}(X), \delta)$.

Further, we will give an example which uses these properties.

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1 Preliminary Facts

We shall present some notions and results used in the sequel (more complete and rigorous treatments may be found in [5], [1]).

Let (X, d) a metric space and $\mathcal{K}(X)$ be the class of all non-empty compact subsets of X.

For $x \in X,\ A \in \mathcal{K}(X)$, we define the *distance* between x and A by

$$\mathrm{d}(x, A) = \inf \left\{ \mathrm{d}(x, a) :\ a \in A \right\}.$$

We also define *the open ball centered at A of radius* $\varepsilon > 0$ by

$$\mathrm{B}(A, \varepsilon) = \{x \in X : \ \mathrm{d}(x, A) < \varepsilon\}.$$

For every $A, B \in \mathcal{K}(X)$, we will denote

$$\mathrm{d}(A, B) = \sup_{x \in A} \mathrm{d}(x, B) = \sup_{x \in A} (\inf_{y \in B} \mathrm{d}(x, y)).$$

The function $\delta : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$

$$\delta(A, B) = \max \{\mathrm{d}(A, B), \mathrm{d}(B, A)\}$$

defines a metric, called *the Hausdorff-Pompeiu metric.*

It follows immediately that $\delta(A, B) < \varepsilon$ if and only if

$$A \subset \mathrm{B}(B, \varepsilon) \text{ and } B \subset \mathrm{B}(A, \varepsilon).$$

2 Some convergence properties in $(\mathcal{K}(X), \delta)$

Theorem 1 *Let (X, d) be a metric space and $(A_n)_{n \geq 1}$ be a sequence of compact subsets of X. Then:*

α) if $A_n \subset A_{n+1}$, for all $n \in \mathbb{N}^*$, and the set $A := \bigcup_{n=1}^{\infty} A_n$ is relatively compact, then

$$\overline{A} = \overline{\bigcup_{n \geq 1} A_n} = \lim_n A_n;$$

β) if $A_{n+1} \subset A_n, \ \forall \, n \in \mathbb{N}^*$, then

$$\lim_n A_n = \bigcap_{n=1}^{\infty} A_n;$$

γ) *if $K \subset X$ is compact set, $A_n \subset K$ for all $n \geq 1$, and $\lim\limits_n A_n = K$, then*

$$K = \overline{\bigcup_{n\geq 1} A_n};$$

δ) *if there exists a non-empty compact set K such that $K \subset A_n$, for all $n \geq 1$, and suppose that $\lim\limits_n A_n = K$, then*

$$K = \bigcap_{n=1}^{\infty} A_n,$$

the limits are taken with respect to the Hausdorff-Pompeiu metric and the bar means the closure.

Proof: α) Let $\varepsilon > 0$. We will find $N \in \mathbb{N}$ such that, for each $n \geq N$, we have

$$\mathrm{d}(\bar{A}, A_n) = \sup_{b\in\bar{A}}(\inf_{a\in A_n} \mathrm{d}(b, a)) < \varepsilon.$$

Since $n \geq N$ implies $A_N \subset A_n$ and hence $\mathrm{d}(\bar{A}, A_n) \leq \mathrm{d}(\bar{A}, A_N)$, it suffices to find $N \in \mathbb{N}$ such that

$$\forall\, b \in \bar{A},\ \exists\, x \in A_N \text{ with } \mathrm{d}(b, x) < \varepsilon.$$

Indeed, if:

$$\forall\, n \in \mathbb{N},\ \exists\, b_n \in \bar{A} \text{ such that} \tag{1}$$

$$\text{for all } x \in A_n,\ \text{ we have } \mathrm{d}(b_n, x) \geq \varepsilon.$$

Since $\bar{A}$ is a compact set, there exists a convergent, hence Cauchy, subsequence $(b_{n_k})_k$ of $(b_n)_n$. So there exists $k_0 \in \mathbb{N}$ such that

$$\text{for each } l \geq k_0, \text{ we have } \mathrm{d}(b_{n_{k_0}}, b_{n_l}) < \frac{\varepsilon}{2}, \text{ and} \tag{2}$$

$$\text{if } b_{n_{k_0}} \in \bar{A} \text{ then } \exists\,(a_p^{k_0})_p \subset A,\ a_p^{k_0} \xrightarrow{p} b_{n_{k_0}}.$$

Let $p \in \mathbb{N}$ such that

$$\mathrm{d}(a_p^{k_0}, b_{n_{k_0}}) < \frac{\varepsilon}{2}. \tag{3}$$

Let $l \geq k_0$ such that $a_p^{k_0} \in A_{n_l}$.

Thus, using relations (1), (2) and (3), we obtain

$$\varepsilon \leq \mathrm{d}(b_{n_l}, a_p^{k_0}) \leq \mathrm{d}(b_{n_l}, b_{n_{k_0}}) + \mathrm{d}(b_{n_{k_0}}, a_p^{k_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is a contradiction.

Because $A_n \subset \bar{A}$ implies $\mathrm{d}(A_n, \bar{A}) = 0$, we have

$$\delta(A_n, \bar{A}) < \varepsilon,\ \forall\, n \geq N.$$

β) Since $\bigcap_{k=1}^{\infty} A_k \subset A_n$, it follows $\mathrm{d}(\bigcap_{k=1}^{\infty} A_k, A_n) = 0$ for all n.

We shall show that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that, for each $n \geq N$, one has

$$\mathrm{d}(A_n, \bigcap_{k=1}^{\infty} A_k) < \varepsilon,$$

namely for all $n \geq N$, and all $x \in A_n$ it follows

$$\exists\, y \in \bigcap_{k=1}^{\infty} A_k \text{ with } \mathrm{d}(x, y) < \varepsilon.$$

Indeed, assume that there exists $\varepsilon > 0$ such that

$$\forall\, n \in \mathbb{N},\ \exists\, x_n \in A_n \text{ so that} \tag{4}$$

$$\forall\, y \in \bigcap_{k=1}^{\infty} A_k \text{ we have } \mathrm{d}(x_n, y) \geq \varepsilon.$$

The sequence $(x_n)_n$ contains a subsequence $(x_{n_p})_p$ which is convergent to x. Since $(A_n)_n$ is a decreasing sequence of sets, we deduce that $\bigcap_{k=1}^{\infty} A_k = \bigcap_{p=1}^{\infty} A_{n_p}$. It is simple to prove that $x \in \bigcap_{p=1}^{\infty} A_{n_p}$.

Thus, it follows that $x \in \bigcap_{k=1}^{\infty} A_k$.

Next, taking $y = x$, we deduce that $\mathrm{d}(x_n, y) < \varepsilon$, for sufficiently large n. This contradicts (4).

γ) Since $A_n \subset K$, $\forall\, n$ it follows that $\bigcup_{n=1}^{\infty} A_n \subset K$ and, also, that $\overline{\bigcup_{n\geq 1} A_n} \subset K$.

Now, let $x_0 \in K$ and suppose that

$$\exists\, \varepsilon > 0 \text{ such that } \mathrm{B}(x_0, \varepsilon) \cap \bigcup_{n=1}^{\infty} A_n = \emptyset. \tag{5}$$

As $A_n \longrightarrow K$, it follows that there exists $N(\varepsilon) \in \mathbb{N}$ such that, if $n \geq N(\varepsilon)$, then $\delta(K, A_n) < \varepsilon$, hence $\mathrm{d}(K, A_n) < \varepsilon$. Thus

$$\sup_{x\in K}\big(\inf_{y\in A_n} \mathrm{d}(x, y)\big) < \varepsilon.$$

Let $n \geq N(\varepsilon)$. Then, there is $y \in A_n$ such that $\mathrm{d}(x_0, y) < \varepsilon$. This contradicts (5).

Thus, $x_0 \in \overline{\bigcup_{n\geq 1} A_n}$.

δ) The inclusion $K \subset \bigcap_{n=1}^{\infty} A_n$ is obvious.

Let $x_0 \in \bigcap_{n=1}^{\infty} A_n$ and suppose that there exists $\varepsilon > 0$ such that

$$\mathrm{B}(x_0, \varepsilon) \cap K = \emptyset.$$

Hence $\inf_{y\in K} \mathrm{d}(x_0, y) \geq \varepsilon$.

It follows that

$$\sup_{x\in A_n} \big(\inf_{y\in K} \mathrm{d}(x, y)\big) \geq \varepsilon,$$

for all $n \geq 1$.

This contradicts the fact that $\lim_n A_n = K$.

□

Now we describe a practical application of above theorem.

Let $X, Y \neq \emptyset$ and $\mathcal{S} = (\omega_i)_{i\in\Im}$ a system of functions $\omega_i : X \longrightarrow Y$, $i \in \Im$. We say that a non-empty set $A \subset X$ is *invariant* for the family $\mathcal{S}$ if

$$A = \mathcal{S}(A) := \bigcup_{i\in\Im} \omega_i(A).$$

Theorem 2 *Let (X, d) be a metric space and $\mathcal{S} = (\omega_n)_{n\geq 1}$ be a sequence of functions $\omega_n : X \longrightarrow X$. We assume that, for each $k \geq 1$, there exists a compact invariant set A_k for the* *family* $(\omega_n)_{n=1}^k$ *and that* $A_k \subset A_{k+1}$. *We suppose, further, that the set* $A := \bigcup_{k=1}^{\infty} A_k$ *is relatively compact.*

Then A *is invariant for* $\mathcal{S}$ *and*

$$\overline{A} = \lim_k A_k$$

the limit being taken in $(\mathcal{K}(X), \delta)$.

Proof: First we will demonstrate the inclusion

$$\bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k) \subset \bigcup_{k=1}^{\infty} A_k. \tag{6}$$

Indeed, let $x \in \bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k)$. Then there are $k \geq 1$ and $n \geq k+1$ such that $x \in \omega_n(A_k)$.

By using the hypothesis, we deduce that $A_k \subset A_n$, hence

$$\omega_n(A_k) \subset \omega_n(A_n) \subset A_n$$

.

So $x \in A_n$, hence $x \in \bigcup_{k=1}^{\infty} A_k$.

Then, using the hypothesis, we have

$$\mathcal{S}(\bigcup_{k=1}^{\infty} A_k) = \bigcup_{n=1}^{\infty} \omega_n(\bigcup_{k=1}^{\infty} A_k) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \omega_n(A_k) =$$

$$= \bigcup_{k=1}^{\infty} \left(\bigcup_{n=1}^{k} \omega_n(A_k) \cup \bigcup_{n=k+1}^{\infty} \omega_n(A_k) \right) =$$

$$= \bigcup_{k=1}^{\infty}\left(A_k \cup \bigcup_{n=k+1}^{\infty} \omega_n(A_k)\right) = \bigcup_{k=1}^{\infty} A_k \cup \bigcup_{k=1}^{\infty}\bigcup_{n=k+1}^{\infty} \omega_n(A_k) = \bigcup_{k=1}^{\infty} A_k$$

where, in the last equality we used the inclusion (6).

Thus, we deduce that $A = \bigcup_{k=1}^{\infty} A_k$ is invariant for $\mathcal{S}$.

The last assertion follows by Theorem 1.

$\square$

References

[1] M. F. Barnsley, *Fractals everywhere*, Academic Press, Harcourt Brace Janovitch, 1988

[2] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, **85**, 1985

[3] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969

[4] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. J. Math. **30**, 1981 (p.713-747)

[5] V. I. Istrăţescu, *Introducere în teoria punctelor fixe*, Ed. Romanian Academy, Bucharest, 1973

[6] C. A. Rogers, *Hausdorff Measure*, Cambridge University Press, 1970

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