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The existance of the solution for some integral equations

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Abstract

The aim of this article is to prove the existance of the positive and continuous solution for one certain type of integral equation. The proof that follows is based on a very well known result, the Schauder's fixed point theorem and the Lemma 1.

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1 Introduction

In the following we will use the next results:

Lemma 1 *Let X be a Banach space, $x^* \in X^*$, $x^* \neq 0$, $H = \{x \in X : x^*(x) = 1\}$ and $A : H \longrightarrow X$ a continuous and compact application. Let it be proved that if*

$$(i) \qquad x^*(A(x)) > 0 \qquad \forall\, x \in H$$

(ii) exists $L > 0$, *with* $\| A(x) \| \leq Lx^*(A(x)) \quad \forall\, x \in H$, *then there exists at least one proper positive value for* A, *that is that there exists* $\lambda_o > 0$ *and there exists* $x_0 \in H$, *with* $A(x_0) = \lambda_0 x_0$

See [3].

Theorem 1 (*J.Schauder, 1930*) *Let* X *be a complete Hausdorff local convex space,* $Z \subset X$ *non-empty, convex, closed,* $A : Z \to Z$ *continue with A(z) relative compact. Then* A *has a fixed point in Z.*

See [1].

2 Main results

Theorem 2 *Let* $K(s,t)$ *be a positive function continuous, defined for* $s,t \in [0,1]$ *with* $K(s,t) > 0$ *for any* $s \in [0,1]$. *Let* $\alpha > 0$. *Let it be proved that there exists one positive continuous function* $v(t)$, *that satisfies the integral equation :*

$$v(s) = \int_0^1 \frac{K(s,t)}{[v(t)]^\alpha} dt \qquad s \in [0,1] \tag{1}$$

Proof. : Let $\varepsilon > 0$ and $K_\varepsilon(s,t) = K(s,t) + \varepsilon$

Applying lemma 1, with $X=[0,1]$, $x^*(x)=\int\limits_0^1 x(t)dt$,

$$A(x)(s)=\int\limits_0^1 \frac{K_\varepsilon(s,t)}{[x(t)]^\alpha}dt,$$

we have

$$x^*(A_x)=\int\limits_0^1\int\limits_0^1 K_\varepsilon(s,t)[x(t)]^\alpha dt\ ds>0.$$

Then, we put

$$L=\max_{s,t\in[0,1]}\frac{K_\varepsilon(s,t)}{\int\limits_0^1 K_\varepsilon(s,t)ds}>0$$

for any $x\in H=\{x\in C[0,1]\ : x^*(x)=0\}$ we find

$$\| A(x) \|=\max_{s\in[0,1]}\int\limits_0^1 \frac{K_\varepsilon(s,t)}{[x(t)]^\alpha}dt=$$

$$=\max_{s\in[0,1]}\int\limits_0^1 [\frac{K_\varepsilon(s,t)}{\int\limits_0^1 K_\varepsilon(s,t)ds}\cdot\frac{1}{[x(t)]^\alpha}\int\limits_0^1 K_\varepsilon(s,t)ds]dt\le$$

$$\le\max_{s\in[0,1]}\int\limits_0^1 [\max_{t\in[0,1]}\frac{K_\varepsilon(s,t)}{\int\limits_0^1 K_\varepsilon(s,t)ds}\int\limits_0^1 \frac{K_\varepsilon(s,t)}{[x(t)]^\alpha}ds]dt\le$$

$$\le L\int\limits_0^1\int\limits_0^1 \frac{K_\varepsilon(s,t)}{[x(t)]^\alpha}dsdt=Lx^*A(x)$$

Based on lemma 1, exists $x_\varepsilon > 0$ and exists $x_\varepsilon \in H$, with $A(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon$, that is

$$\int_0^1 \frac{K_\varepsilon(s,t)}{[x_\varepsilon(t)]^\alpha} dt = \lambda_\varepsilon x_\varepsilon(s), \quad s \in [0,1].$$

With the notation $v_\varepsilon(t) = \lambda_\varepsilon^{\frac{1}{1+\varepsilon}} \cdot x_\varepsilon(t)$, the last equality becomes

$$(2) \qquad \int_0^1 \frac{K_\varepsilon(s,t)}{[v_\varepsilon(t)]^\alpha} dt = v_\varepsilon(s), \quad s \in [0,1]$$

Multiplying (2) by $\dfrac{1}{[v_\varepsilon(s)]^{\alpha+1}}$ and integrating, we obtain

$$\int_0^1\int_0^1 K_\varepsilon(s,t) \frac{1}{[v_\varepsilon(t)]^\alpha} \cdot \frac{1}{[v_\varepsilon(s)]^{\alpha+1}} dt ds = \int_0^1 \frac{1}{[v_\varepsilon(s)]^\alpha} ds$$

Since $K(s,s) > 0 \quad \forall\, s \in [0,1]$, $\exists\, \delta > 0$ so that $K(s,t) \geq \delta$ for any $s,t \in [0,1]$ with $\mid t - s \mid \leq 2\delta$. That's why, for any $\varepsilon \in [0, 1-\delta]$ we find

$$(3) \qquad \delta \left(\int_\xi^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \right) \left(\int_\xi^{\xi+\delta} \frac{ds}{[v_\varepsilon(s)]^{\alpha+1}} \right) \leq \int_0^1 \frac{ds}{[v_\varepsilon(s)]^\alpha}.$$

From the Holder's equality we deduce

$$\int_\xi^{\xi+\delta} \left(1 \cdot \frac{1}{[v_\varepsilon(t)]^\alpha} \right) dt \leq \left(\int_\xi^{\xi+\delta} 1 \cdot dt \right)^{\frac{1}{\alpha+1}} \left(\int_\xi^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^{\alpha+1}} \right)^{\frac{\alpha}{\alpha+1}},$$

from where

$$\int_{\xi}^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \le \delta^{\frac{1}{1+\alpha}} \left(\int_{\xi}^{\xi+\alpha} \frac{ds}{[v_\varepsilon(s)]^{\alpha+1}} \right)^{\frac{\alpha}{\alpha+1}}.$$

Results that

$$\delta^{\frac{-1}{\alpha}} \left(\int_{\xi}^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \right)^{1+\frac{1}{\alpha}} \le \int_{\xi}^{\xi+\delta} \frac{ds}{[v_\varepsilon(s)]^\alpha}.$$

From here and from (3) we obtain that for any $\xi \in [0, 1-\delta]$, takes place the inequality

$$\delta^{1-\frac{1}{\alpha}} \left(\int_{\xi}^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \right)^{2+\frac{1}{\alpha}} \le \int_{0}^{1} \frac{ds}{[v_\varepsilon(s)]^\alpha}$$

so

$$\int_{\xi}^{\xi+\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \le \alpha^{-\frac{1-\frac{1}{\alpha}}{2+\frac{1}{\alpha}}} \left(\int_{0}^{1} \frac{dt}{[v_\varepsilon(t)]^\alpha} \right)^{\frac{1}{2+\frac{1}{\alpha}}}$$

Admitting that $n\delta = 1$, with $n \in N$, we deduce

$$\int_{0}^{1} \frac{dt}{[v_\varepsilon(t)]^\alpha} = \sum_{k=1}^{n} \int_{(k-1)\delta}^{k\delta} \frac{dt}{[v_\varepsilon(t)]^\alpha} \le n\delta^{-\frac{1-\frac{1}{\alpha}}{2+\frac{1}{\alpha}}} \left(\int_{0}^{1} \frac{dt}{[v_\varepsilon(t)]^\alpha} \right)^{\frac{1}{2+\frac{1}{\alpha}}}$$

from where

$$\int_{0}^{1} \frac{dt}{[v_\varepsilon(t)]^\alpha} \le C \tag{4}$$

with C=const., $c > 0$, C independent from ε.

From (2) and (4) we have

$$v_\varepsilon(s) \leq \frac{K_0}{C}, \forall\ \varepsilon > 0, \forall\ s \in [0,1]$$

where $K_0 = \max\{K(s,t); s,t \in [0,1]\}$.

For any $s \in [0,1]$ and any $t \in [0,1]$, with $\mid t-s \mid \leq 2\delta$, we have $K(s,t) \geq \delta$ and (2) we obtain

$$v_\varepsilon(s) = \int\limits_0^1 \frac{K_\varepsilon(s,t)}{[v_\varepsilon(t)]^\alpha} dt \geq \int\limits_s^{s+\delta} \frac{\delta}{(\frac{K_0}{C})^\alpha} dt = \frac{C^\alpha \delta^2}{K_0^\alpha}$$

From these inequalities result that $(v_\varepsilon(s))_{\varepsilon>0}$ is an uniform bounded familly and equal uniform continue. Based on Ascoli's theorem [2], $(v_{\frac{1}{k}}(s))_{k\in N}$.

Contains a subsequence convergent to a function v(s). Passing to the limit in (2), we obtain (1) in which obviously $v_0 \in C[0,1]$ and $v(s) > 0$ for any $s \in [0,1]$.

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