General Mathematics Vol. 8, No. 1–2 (2000)

Note on a Delta Operator

Emil C. Popa

Abstract

In the end of the paper "Finite operator calculus" by Gian-Carlo Rota, D. Kahaner, A. Odlyzko [3] are raised a set to find formulas for $p_n(Q)$ where (p_n) is a Sheffer sequence relative to the delta-operator Q. In this paper we consider the Abel polynomials

$$a_n(x) = x(x+nb)^{n-1}\ , \qquad n = 1, 2, ...$$

and we study some umbral properties for delta operator

(1) $$a_n(Q) = Q(Q+nbI)^{n-1}\ , \qquad b \neq 0\ , \qquad n \text{ fixed}$$

defined on the algebra of polynomials, where Q is a delta operator.

1991 Mathematics Subject Classification: 05A40

1°. Taking into account that

$$(D+nbI)^{n-1} = e^{-nbx}D^{n-1}e^{nbx}$$

(see [3]) we find the formula

(2) $$a_n(D) = De^{-nbx}D^{n-1}e^{nbx}$$

Next we observe that

$$(3)\qquad a_n(Q) = (nb)^n \sum_{k=1}^{n} \frac{1}{nb^k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} Q^k$$

and for $Q = DE^a$ (the Abel operator) we have

$$a_n(Q) = (nb)^n \sum_{k=1}^{n} \frac{a^k}{nb^k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} D^k E^a$$

2°. Now we consider the following :

Theorem 1 *If $a_n(Q)$ is defined as in (1), then*

$$[a_n(Q)]'^{-1} = \frac{1}{(nb)^{n-1}\Gamma(n-2)} \int_0^\infty e^{-t} t^{n-3} (e^{-\frac{1}{nb}tQ} q)(x)dt$$

where

$$q(x) = \int_0^\infty e^{-s} (e^{-\frac{1}{b}sQ} Q'^{-1} p)(x)ds$$

$n = 2, 3, \ldots,$ *and Q' is the Pincherle derivative.*

Proof: Taking account that

$$(4)\qquad (P_\alpha^n p)(x) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-t} t^{n-1} (e^{\alpha t Q} p)(x)dt$$

where

$$P_\alpha = \frac{I}{I - \alpha Q}$$

and

$$[a_n(Q)]' = n(Q + nbI)^{n-2}(Q + bI)Q'$$

we obtain

$$[a_n(Q)]'^{-1} = \frac{1}{(nb)^{n-1}} \cdot \frac{I}{(I + \frac{1}{nb}Q)^{n-2}} \cdot \frac{I}{I + \frac{1}{b}Q} Q'^{-1} =$$

$$= \frac{1}{(nb)^{n-1}\Gamma(n-2)} \int_0^\infty e^{-t} t^{n-3} (e^{-\frac{1}{nb}tQ})(x) dt$$

where

$$q(x) = \int_0^\infty e^{-s}(e^{-\frac{1}{b}sQ} Q'^{-1} p)(x) ds$$

3°. Now we note

$$a_n(D) = D(D + nbI)^{n-1}$$

where D is the derivative operator $(Dp)(x) = p'(x)$, and let P by the operator defined by

$$P = (D + nbI)^{n-1}.$$

If

$$s_m = P^{-m+1} e_m, s_0 = Pe_0 = c \neq 0$$

where $e_m = x^m, m = 0, 1, 2, ...$

We have

$$a_n(D) s_m = mP^{-m+2} e_{m-1} = m s_{m-1}$$

and (S_n) is a Sheffer set relative to $a_n(D)$.

Using (2) we find the integral representation

$$s_m = \frac{1}{(nb)^{(m-1)(n-1)}} \cdot \frac{I}{(I + \frac{1}{nb}D)^{(m-1)(n-1)}} e_m = \tag{5}$$

$$= \frac{1}{(nb)^{(m-1)(n-1)}\Gamma(mn-m-n+1)} \int_0^\infty e^{-t}\, t^{mn-m-n} \left(x - \frac{1}{nb}t\right)^m dt.$$

$\square$

References

[1] S.Roman , *The umbral calculus* , Academic Press, New York, 1984.

[2] S.Roman ,G.-C.Rota *The umbral calculus* , Adv. in Math.27, 1978, 95-188.

[3] G.-C.Rota , D.Kahaner , A.Odlyzo , *Finite Operator Calculus* , Journal of Mathematical Analysis and Applications , 42(1973), 685-760.

Department of Mathematics
University "Lucian Blaga" of Sibiu
Str.Dr.I.Raţiu, No.7
2400 Romania