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Some preserving properties of certain integral operators

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Abstract

In this article we show that the class of uniform starlike functions with negative coefficients and the class of uniform convex functions with negative coefficients are preserved by the Alexander integral operator and the class of uniform convex functions with negative coefficients is preserved by the Bernardi integral operator.

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1 Introduction

Let A be the set of all analytic functions f in the unit disc U, with $f(0) = 0, f'(0) = 1$ and $S = \{f \in A, F \in H_u(U)\}$ is the set of analytic and univalent functions in the unit disc U.

Let define the Alexander integral operator $I_A : A \to A$:

$$(1) \qquad f(z) = I_A F(z) = \int_0^z \frac{F(t)}{t} dt \quad , z \in U$$

and the Bernardi integral operator $I_a : A \to H(U)$:

$$(2) \quad f(z) = I_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \quad , a = 1, 2, 3, ...$$

The purpose of this note is to show that the class of uniform starlike functions with negative coefficients and the class of uniform convex functions with negative coefficients are preserved by the Alexander integral operator and the class of uniform convex functions with negative coefficients is preserved by the Bernardi integral operator.

2 Preliminary results

Definition 1 *Let T be the set of all functions $f \in S$ having the form:*

$$(3) \quad f(z) = z - \sum_{j=2}^{\infty} a_j \cdot z^j \ , \ a_j \geq 0 \ , \ j = 2, 3, ... \ \ z \in U$$

Theorem 1 *See[5]. If $f \in T$, then the next two assertions are equivalent:*

$$(i) \qquad \sum_{j=2}^{\infty} j \cdot a_j \leq 1$$

$$(ii) \qquad f \in T$$

.

Definition 2 *See[2]. A function f is said to be uniformly starlike in the unit disc U if f is starlike and has the property that for every circular arc γ contained in U, with center α also in U, the arc $f(\gamma)$ is starlike with respect to $f(\alpha)$. We let US^* denote the class of all such functions. By taking $UT^* = T \cap US^*$ we define the class of uniformly starlike functions with negative coefficients.*

Remark 1 *An arc $f(\gamma)$ is starlike with respect to a point $w_0 = f(\alpha)$ if $arg(f(z) - w_0)$ is nondecreasing as z traces γ in the positive direction.*

Theorem 2 *See[3]. Let $f \in S, f(z) = z + \sum\limits_{n=2}^{\infty} a_n \cdot z^n, a_n \in C$. If $\sum\limits_{n=2}^{\infty} n \cdot |a_n| \leq \frac{\sqrt{2}}{2}$ then $f \in US^*$.*

Definition 3 *See[1]. A function f is uniformly convex in the disc U if $f \in S^C$ (convex) and for every circular arc γ contained in U, with center α also in U , the arc $f(\gamma)$ is also convex. We let US^C denote the class of all such functions. By taking $UT^C = T \cap US^C$ we define the class of uniformly convex functions with negative coefficients.*

Remark 2 *The arc $\gamma(t)$, $a < t < b$ is convex if the argument of the tangent to $\gamma(t)$ is nondecreasing with respect to t.*

Theorem 3 *See[6]. Let* $f \in T, f(z) = z - \sum_{n=2}^{\infty} a_n \cdot z^n$. *Then* $f \in UT^C$ *if and only if :*

$$\sum_{n=2}^{\infty} n(2n-1) \cdot a_n \leq 1.$$

Theorem 4 *See[6].* $T^C \subset UT^* \subset T^*$ *where*
$T^C = T \cap S^C \quad , \ T^* = T \cap S^*$.

Remark 3 *In [4] the author showed that the Libera integral operator* $f : A \to A$ *defined as* $f(z) = I_1 F(z) = \frac{2}{z} \int_0^z F(t)dt \quad ,$ $z \in U$ *preserve the class* UT^*.

3 Main results

Theorem 5 *The Alexander integral operator defined by (1) preserves the class* UT^*, *that is :*

$$If \quad F(z) \in UT^*, \quad then \quad f(z) = I_A F(z) \in UT^*.$$

Proof. Let $F(z) \in UT^*$, $F(z) = z - \sum_{n=2}^{\infty} a_n \cdot z^n \quad , a_n \geq 0$.

From Theorem 4 we have $UT^* \subset T^*$, and thus $F \in T^*$.

From Theorem 1 we have

(4)
$$\sum_{n=2}^{\infty} n \cdot a_n \leq 1$$

Then $f(z) = I_A F(z) = \int_0^z \frac{F(t)}{t} dt = \int_0^z \frac{1}{t} \left(t - \sum_{n=2}^{\infty} a_n \cdot t^n \right) dt =$
$= z - \sum_{n=2}^{\infty} b_n \cdot z^n \quad \text{with} \quad b_n = \frac{a_n}{n} \geq 0.$

It follows that $f \in T$. We have now the proof that $f \in US^*$.

By using Theorem 2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n \cdot |b_n| \leq \frac{\sqrt{2}}{2},$$

that is:

$$\sum_{n=2}^{\infty} \sqrt{2} \cdot n \cdot b_n \leq 1$$

Because $\sqrt{2} \cdot n \cdot b_n = \sqrt{2} \cdot a_n \leq n \cdot a_n \quad , n \geq 2$ and from (4), we obtain: $\sum\limits_{n=2}^{\infty} \sqrt{2} \cdot n \cdot b_n \leq 1$, and thus, $f \in US^*$.

From $f \in T$ and $f \in US^*$ we obtain that $f \in UT^*$.

Theorem 6 *The Alexander integral operator defined by (1) preserves the class UT^C, that is:*
if $F(z) \in UT^C$, then $f(z) = I_A F(z) \in UT^C$

Proof. Let $F(z) \in UT^C$, $F(z) = z - \sum\limits_{n=2}^{\infty} a_n \cdot z^n \quad , a_n \geq 0$.

From Theorem 3 we have

$$\sum_{n=2}^{\infty} n(2n-1) \cdot a_n \leq 1 \tag{5}$$

Thus,

$$f(z) = I_A F(z) = \int_0^z \frac{F(t)}{t} dt = \int_0^z \frac{1}{t} \left(t - \sum_{n=2}^{\infty} a_n \cdot t^n \right) dt =$$

$$= z - \sum_{n=2}^{\infty} b_n \cdot z^n \quad cu \quad b_n = \frac{a_n}{n} \geq 0$$

We obtain that $f \in T$. Then, by using Theorem 3 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(2n-1) \cdot b_n \leq 1.$$

From $b_n \leq \frac{a_n}{n} \leq a_n$, $n \geq 2$ and from (5), we obtain: $\sum_{n=2}^{\infty} n(2n-1) \cdot b_n \leq 1$, and thus, $f \in UT^C$.

Theorem 7 *The Bernardi integral operator defined by (2) preserves the class UT^C, that is:*

$$If \quad F(z) \in UT^C \quad , \quad then \quad f(z) = I_a F(z) \ \in UT^C.$$

Proof. Let $F(z) \in UT^C$, $F(z) = z - \sum_{n=2}^{\infty} a_n \cdot z^n$, $a_n \geq 0$.

From Theorem 3 we have

$$\sum_{n=2}^{\infty} n(2n-1) \cdot a_n \leq 1 \tag{6}$$

Thus

$$f(z) = I_a F(z) = \frac{1+a}{z^a} \cdot \int_0^z F(t) \cdot t^{a-1} dt =$$

$$= \frac{1+a}{z^a} \int_0^z \left(t - \sum_{n=2}^{\infty} a_n \cdot t^n \right) \cdot t^{a-1} dt = z - \sum_{n=2}^{\infty} b_n \cdot z^n,$$

where

$$b_n = \frac{(a+1) \cdot a_n}{a+n} \geq 0 \quad (a = 1, 2, ...)$$

We obtain that $f \in T$. By using Theorem 3 it is sufficient to prove that:

$$\sum_{n=2}^{\infty} n(2n-1) \cdot b_n \leq 1.$$

From $\frac{(a+1)}{a+n} \cdot a_n \leq a_n$ for $n \geq 2$ and from (6), we obtain

$$\sum_{n=2}^{\infty} n(2n-1) \cdot b_n \leq 1,$$

and thus, $f \in UT^C$.

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