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A special class of univalent starlike functions

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Abstract

The purpose of this paper is to introduce a subclass of univalent starlike functions f, defined by means of the starlike function

$$F(z) = f(z)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1-\alpha+\alpha\frac{zf'(z)}{f(z)}\right)\right]^{\frac{1}{2}}.$$

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1 Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbf{C},\ |z| < 1\},$$

let $\mathcal{H}[U]$ be the space of holomorphic functions in U.

We let:

$$A_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots,\ z \in U\}$$

with $A_1 = A$,

$$S^* = \left\{ f \in A,\ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$

the class of starlike functions in U,

$$M_\alpha = \left\{ f \in A,\ \mathrm{Re}\,\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f(z)} + 1 \right) \right] > 0,\ z \in U \right\}$$

the class of α-convex functions,

$$S^*(\alpha) = \left\{ f \in A,\ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\}$$

the class of starlike functions of order α, $0 \le \alpha < 1$.

Theorem 1. See [2] *If $\alpha \in \mathbf{R}$ and $f \in M_\alpha$, then f is a starlike function.*

Theorem 2. See [2] *If $\alpha \ge 0$ and $f \in A$, then $f \in M_\alpha$ if and only if*

$$F(z) = f(z)\left[\frac{zf'(z)}{f(z)} \right]^\alpha$$

is starlike.

Let $\alpha \in \mathbf{C}$ and $f \in A$. We say that the function f is α-starlike if the function $F : U \to \mathbf{C}$, where

$$F(z) = f(z)\left(1 - \alpha + \alpha\frac{zf'(z)}{f(z)} \right), \quad z \in U$$

is starlike [4].

Lemma. See [1] *Let $\psi : \mathbf{C}^2 \to \mathbf{C}$, satisfy the condition*

$$\mathrm{Re}\,\psi(is, t) \le 0,$$

for $s \in \mathbf{R}$, $t \leq -\frac{1}{2}(1+s^2)$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ *satisfies*

$$\mathrm{Re}\, \psi[p(z), zp'(z)] > 0,$$

then

$$\mathrm{Re}\, p(z) > 0.$$

More general forms of this lemma can be found in [1].

2 Main results

Definition. Let $\alpha \in \mathbf{R}$ and let $f \in A$, with

$$\frac{f(z)f'(z)}{z} \neq 0, \quad 1 - \alpha + \alpha\frac{zf'(z)}{f(z)} \neq 0, \quad z \in U.$$

We say that the function f belongs to the class $M_{\alpha,\frac{1}{2}}$ if the function $F : U \to \mathbf{C}$ defined by

$$F(z) = f(z)\left[\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 - \alpha + \alpha\frac{zf'(z)}{f(z)}\right)\right]^{\frac{1}{2}} \tag{1}$$

is starlike in U.

Theorem. *For any real number* α *which satisfies the condition* $\alpha(1-\alpha) \geq 0$, *we have the inclusion* $M_{\alpha,\frac{1}{2}} \subset S^*$.

Proof. If we let $p(z) = \dfrac{zf'(z)}{f(z)}$, $z \in U$, then from the condition $\dfrac{f(z)f'(z)}{z} \neq 0$, we deduce that the function p is holomorphic in U.

Since F is starlike we have

$$\text{Re}\,\frac{zF'(z)}{F(z)} > 0, \quad z \in U \tag{2}$$

and from (1) we deduce

$$\frac{zF'(z)}{F(z)} = p(z) + \frac{\alpha}{2}\cdot\frac{zp'(z)}{p(z)} + \frac{\alpha}{2}\cdot\frac{zp'(z)}{1-\alpha+\alpha p(z)} = $$
$$= \psi[p(z), zp'(z)].$$

Hence condition (2) is equivalent to

$$\text{Re}\,\psi[p(z), zp'(z)] > 0.$$

Since

$$\text{Re}\,\psi(is,t) = \text{Re}\,\left[is + \frac{\alpha}{2}\cdot\frac{t}{is} + \frac{\alpha}{2}\cdot\frac{t}{1-\alpha+\alpha is}\right] =$$
$$= \frac{\alpha}{2}t\frac{1-\alpha}{(1-\alpha)^2+\alpha^2 s^2} \le 0,$$

for all real s and $t \le -\frac{1}{2}(1+s^2)$, by using the Lemma we deduce that $\text{Re}\,p(z) > 0$ which shows that $M_{\alpha,\frac{1}{2}} \subset S^*$. $\square$

References

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