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Note on some basic polynomials

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Abstract

In this paper we study some properties of the subgroup $\mathcal{B}$ of the binomial polynomials.

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1°. Let $P = (p_n(x))$ be the basic sequence for some delta operator Q, and

$$\mathcal{P} = \{A = (a_n(x) \ / \ a_0(x) \neq 0, \ \deg a_n = n, \ n = 0, 1, 2, \ldots)\}.$$

On defines the Q–umbral product of the sequences A and B

$$A, B \in \mathcal{P} \quad , \quad A = (a_n(x)) \quad , \quad B = (b_n(x))$$

$$A \overset{Q}{\otimes} B = C \quad , \quad C = (c_n(x))$$

where

$$c_n(x) = \sum_{k=0}^{n} \frac{a_k(x)}{k!}(Q^k b_n(x))(0) \ . \tag{1}$$

Evidently $C \in \mathcal{P}$. Moreover $\mathcal{P}$ endowed with the binary operation defined by (1) is a group, namely the Q–umbral group (see [1]).

We denote by $\mathcal{B}$ the set of the binomial sequences. $\mathcal{B}$ is a subgroup of the Q–umbral group $\mathcal{P}_Q$ (see [3]).

Next we note

$$G_Q = \{\tau_A^Q \ / \ A \in \mathcal{P}_Q\}$$

where τ_A^Q is an umbral operator defined by

$$\tau_A^Q p_n = a_n \quad , \quad n = 0, 1, 2, \ldots$$

We observe that G_Q is a group with the binary operation

$$\tau_A^Q \cdot \tau_B^Q = \tau_C^Q$$

where $C = A \overset{Q}{\otimes} B$.

We have

$$\tau_A^Q \tau_P^Q = \tau_P^Q \tau_A^Q = \tau_A^Q$$

and if A' is the symmetric element for the polynomial sequence A in $\mathcal{P}_Q$, then

$$\tau_A^Q \tau_{A'}^Q = \tau_{A'}^Q \tau_A^Q = \tau_P^Q \ .$$

Moreover the function $\varphi : \mathcal{B} \longrightarrow G_Q$ given by $\varphi(A) = \tau_A^Q$ is a group morphism

$$\varphi(A \overset{Q}{\otimes} B) = \tau_C^Q = \tau_A^Q \tau_B^Q = \varphi(A)\varphi(B) \ .$$

2°. We note now the following: is it true that the subgroup $\mathcal{B}$ is a normal subgroup in $\mathcal{P}_Q$?

Let $\mathcal{S}$ be a subgroup in $\mathcal{P}_D$ where D is the derivative operator.

Theorem. *If exists $A = (a_n(x)) \in \mathcal{P}_D$ and $B = (b_n(x)) \in \mathcal{S}$ such that $C_1 = (c_n(x)) \notin \mathcal{S}$ where*

$$(2) \qquad c_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k!} x^k b_n^{(k)}(0) \ , \quad n = 0, 1, 2, \ldots$$

then

$$\mathcal{S} \overset{D}{\otimes} \mathcal{P}_D \neq \mathcal{P}_D \overset{D}{\otimes} \mathcal{S} \ .$$

Proof. We consider

$$A = (a_n(x) \in \mathcal{P}_D) \ , \quad a_n(x) = (n+1)x^n \ , \quad n = 0, 1, 2, \ldots$$

and

$$A' = (a'_n(x)) \in \mathcal{P}_D \ , \quad a'_n(x) = \frac{1}{n+1}x^n \ , \quad n = 0, 1, 2, \ldots$$

It is easy to show

$$A \overset{D}{\otimes} A' = A' \overset{D}{\otimes} A = P_0$$

where $P_0 = (x^n)$, $n = 0, 1, 2, \ldots$.

If $\mathcal{S} \overset{D}{\otimes} \mathcal{P}_D = \mathcal{P}_D \overset{D}{\otimes} \mathcal{S}$, $\mathcal{S} \overset{D}{\otimes} x = x \overset{D}{\otimes} \mathcal{S}$ for any $x \in \mathcal{P}_D$ then

$$A \overset{D}{\otimes} B \overset{D}{\otimes} A' \in \mathcal{S} \ .$$

We note

$$C_1 = A \overset{D}{\otimes} B \overset{D}{\otimes} A' \ , \quad C_1 = (c_n(x)) \ , \quad n = 0, 1, 2, \ldots$$

and we observe that

$$C_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k!} x^k b_n^{(k)}(0) \ , \quad n = 0, 1, 2, \ldots$$

In conclusion $\mathcal{S} \overset{D}{\otimes} \mathcal{P}_D \neq \mathcal{P}_D \overset{D}{\otimes} \mathcal{S}$. $\mathcal{S}$ is not a normal subgroup in $\mathcal{P}_D$.

Corollary. *If $\mathcal{B}$ is the subgroup of binomial polynomials, then $\mathcal{B} \overset{D}{\otimes} \mathcal{P}_D \neq \mathcal{P}_D \overset{D}{\otimes} \mathcal{B}$. $\mathcal{B}$ is not a normal subgroup in $\mathcal{P}_D$.*

Proof. Let $B = (b_n(x))$, $n = 0, 1, 2, \ldots$ the Abel polynomials

$$b_n(x) = x(x - na)^{n-1} \quad , \quad a \neq 0 \; .$$

Using (2) we obtain

$$\begin{aligned}
c_0(x) &= 1 \\
c_1(x) &= x \\
c_2(x) &= x^2 - \frac{4a}{3}x \\
c_3(x) &= x^3 - \frac{9a}{2}x^2 + \frac{9a^2}{2}x
\end{aligned}$$

But

$$c_3(x + y) \neq \sum_{k=0}^{3} \binom{3}{k} c_k(x) c_{3-k}(y)$$

and hence $\mathcal{B} \overset{D}{\otimes} \mathcal{P}_D \neq \mathcal{P} \overset{D}{\otimes} \mathcal{B}$. $\mathcal{B}$ is not a normal subgroup in $\mathcal{P}_D$.

References

[1] Popa C.E., *Contributions to finite operator calculus*, Doctoral thesis, Cluj–Napoca, 1998.

[2] Popa C.E., *On a subgroup of binomial polynomials*, Proceedings of the Third Romanian–German Seminar on Approximation Theory, General Mathematics, Vol.6, 1998, p.81–82.

[3] Popa C.E., *On the sequences of binomial type*, General Mathematics, Vol.7, 1999, p.103–106.

[4] Roman S., *The umbral calculus*, Academic Press, New York, 1984.

[5] Rota G.C., Kahner D., Odlyzko A., *Finite operator calculus*, J. Math. Anal. Appl. 42 (1973), p. 685–760.

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