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On separability and metrisability of weak topologies

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Abstract

In this paper it is presented a theorem dealing with the equivalence between the separability of the strong dual of a real Banach space, E, and the metrisability of the unity closed ball, B_E, by starting from a similar (proved) theorem in the delightful functional analysis book (see [1]). We prove, in addition, through an example, that the hypotheses for the theorem are essential.

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1 Introduction

In [1], in theorem III.25 is presented the equivalence "*A real Banach space E is separable if and only if the unity closed ball of the dual space E' is metrisable related to the weak topology* $\sigma(E', E)$". The aim of this paper is to prove the dual theorem.

Throughout this paper, E is a real Banach space, endowed with the norm $\|\cdot\|$. The result is obtained through topological methods.

Definition 1. *A subset $M \subseteq E, M \neq \emptyset$ is called metrisable related to the topology τ on the space E if there exists a metrics $d : M \times M \to \mathbb{R}_+$ which induces the topology τ on the subset M.*

The strong (normic) dual space of the real Banach space E is the set

$$E' = \{f : E \to \mathbb{R},\ f \text{ is linear and continuous}\} \tag{1}$$

and the unity closed ball of E is denoted to be the set

$$B_E = \{x \in E,\ \|x\| \leq 1\}. \tag{2}$$

Definition 2. *We say that E' is separable (related to the strong topology) if there exists a subset $F \subseteq E$, $F \neq \emptyset$ and such that $cardF = \aleph_0$ and $\overline{F} = E'$.*

We state and prove the main theorem.

2 Main result

Theorem 1. *Let E be a real Banach space. If E' is separable, then B_E is metrisable related to the weak topology $\sigma(E, E')$. And reciprocally, if B_E is metrisable related to the weak topology $\sigma(E, E')$, then E' is separable.*

Proof. Suppose firstly that E' is a separable space. Then, there exists $(g_n)_{n\in\mathbb{N}^*}$ a counterable family of E' and dense in E'. Consider $(g_n)_{n\in\mathbb{N}^*}\cap B_{E'} \overset{\text{not}}{:=} (f_n)_{n\in\mathbb{N}^*}$ such that $(f_n)_{n\in\mathbb{N}^*}$ is a counterable and dense family in $B_{E'}$. In what follows, we define a metrics $d : B_E \times B_E \to \mathbb{R}_+$ on B_E : for $x, y \in B_E$,

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot |f_n(x-y)|.$$

(it is easy to verify the metrics properties for the application d).

We prove now that the metrics d induces the weak topology $\sigma(E,E')$ on B_E.

a). Let $x_0 \in B_E$ and $V \in \mathcal{V}_{\sigma(E,E')}(x_0)$ be arbitrary. We have to check that there exists $r > 0$ such that

$$U := \{x \in B_E,\ d(x,x_0) < r\} \subseteq V.$$

For this purpose, we take V to be

$$V \overset{\text{not}}{:=} \left\{x \in B_E,\ |f_i(x-x_0)| < \varepsilon,\ (\forall)\ i \in \overline{1,k}\right\},$$

where $\varepsilon > 0$, $\|f_i\|_{E'} \leq 1$, $(\forall)\ i \in \overline{1,k}$. Since $\overline{(f_n)_{n\in\mathbb{N}^*}} = B_{E'}$, we obtain that for each $i \in\mathbb{N}^*$ there exists $n_i = n(i) \in\mathbb{N}^*$ such that

$$\|f_{n_i} - f_i\| < \frac{\varepsilon}{4}.$$

Let us consider $r > 0$ verifying the conditions

$$\text{for every } i \in \overline{1,k},\ 2^{n_i} \cdot r < \frac{\varepsilon}{2}$$

and let $x \in U$ be arbitrary. Therefore, $x \in B_E$ and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot |f_n(x - x_0)| < r.$$

But,

$$\frac{1}{2^{n_i}} \cdot |f_{n_i}(x - x_0)| < r, \ (\forall)\ i \in \overline{1,k}$$

and hence,

$$|f_i(x - x_0)| = |f_i(x - x_0) - f_{n_i}(x - x_0)| + |f_{n_i}(x - x_0)| <$$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad (\forall)\ i \in \overline{1,k}.$

b). Let $x_0 \in B_E$ and $r > 0$ be arbitrary. We check that there exists $V \in \mathcal{V}_{\sigma(E,E')}(x_0)$ in B_E such that

$$V \subseteq U := \{x \in B_E,\ d(x, x_0) < r\}.$$

In other words, we are looking for $\varepsilon > 0$ and $k \in \mathbb{N}^*$ such that V having the form

$$V := \left\{x \in B_E,\ |f_i(x - x_0)| < \varepsilon,\ (\forall)\ i \in \overline{1,k}\right\},$$

is inclosed in U.

Let $x \in V$ be arbitrary. We have

$$\begin{aligned} d(x, x_0) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot |f_n(x - x_0)| = \\ &= \sum_{n=1}^{k} \frac{1}{2^n} \cdot |f_n(x - x_0)| + \sum_{n=k+1}^{\infty} \frac{1}{2^n} \cdot |f_n(x - x_0)| < \\ &< \varepsilon + 2 \cdot \sum_{n=1}^{k+1} \frac{1}{2^n} = \varepsilon + 2 \cdot \frac{1}{2^{k+1}} \cdot \frac{1}{1 - \frac{1}{2}} = \varepsilon + \frac{1}{2^{k-1}}. \end{aligned}$$

Hence, by choosing $k := \max\{1, 3 - \log_2 r\}$ and $\varepsilon < \frac{r}{2}$, we get

$$d(x, x_0) < \frac{r}{2} + \frac{r}{2} = r.$$

Let us suppose that B_E is metrisable related to the weak topology $\sigma(E, E')$ and we want to prove that E' is separable.

For this purpose, let $d : B_E \times B_E \to \mathbb{R}_+$ be a metrics on B_E, which induces the weak topology $\sigma(E, E')$. Denote by

$$U_n := \left\{x \in B_E,\ d(x, 0) < \frac{1}{n}\right\}, \text{ for every } n \in \mathbb{N}^*.$$

Then, by using the hypothesis, there exists $V_n \in \mathcal{V}_{\sigma(E,E')}(0)$, V_n with the form

$$V_n := \{x \in E,\ |f(x)| < \varepsilon_n,\ (\forall)\ f \in \Phi_n\},$$

where $\varepsilon_n > 0$, $\Phi_n \subset E'$ is finite and such that

$$V_n \subseteq U_n, \text{ for every } n \in \mathbb{N}^*.$$

We set

$$D := \bigcup_{n=1}^{\infty} \Phi_n \text{ and } F := \operatorname{Space}(D)$$

and we prove that F is dense in E' (for the strong topology).

If this fact would be false then $\overline{F} \neq E'$. So, let us consider $f' \notin \overline{F}$. Since $\overline{F}$ is a closed set, the distance between f' and $\overline{F}$, $d := \inf\left\{\|f' - f\|,\ f' \in \overline{F}\right\}$, is strictly positive.

Let $r > 0$ be such that $r \cdot d > 1$.

Hence $r \cdot f' \notin \overline{F}$, (since else there exists a sequence having the form $\left(\frac{1}{r}f_n\right)_{n\in\mathrm{N}^*} \subset F$, which is convergent to f'). And, for $f \in F$ arbitrary,

$$\|rf' - f\| = r \cdot \left\| f' - \frac{1}{r}f \right\| \geq r \cdot d > 1,$$

because F is a linear space; and passing to inf for $f \in F$, we deduce

$$dist\,(rf', F) \geq r \cdot d > 1,$$

and therefore, $\mathrm{dist}\big(rf', \overline{F}\big) \geq r \cdot d > 1$.

In conclusion, we have found an element $f_0 \in \overline{F}$ such that

$$\mathrm{dist}\left(f_0, \overline{F}\right) \overset{\text{not}}{:=} d_0 > 1.$$

Consider now Θ the linear space generated by $\{f_0\} \cup \overline{F}$ and define the functional $\xi_0 : \Theta \to \mathbb{R}$, $\xi_0\,(\lambda f_0 + f) := \lambda \cdot d_0$, for every $\lambda f_0 + f \in \Theta$, (each element of Θ can be written in an unique way like this !). It is obviously that ξ_0 is a linear functional and is continuous since:

$$\begin{aligned} |\xi_0\,(\lambda f_0 + f)| &= |\lambda \cdot d_0| = |\lambda| \cdot d_0 \cdot \frac{\left\| f_0 - \frac{1}{-\lambda}f \right\|}{\left\| f_0 - \frac{1}{-\lambda}f \right\|} \leq \\ \leq d_0 \cdot \frac{\|\lambda f_0 + f\|}{\left\| f_0 - \frac{1}{-\lambda}f \right\|} &\leq \|\lambda f_0 + f\|\,, \text{ for } \lambda \neq 0, f \in \overline{F}; \end{aligned}$$

this inequality is true in the case $\lambda = 0$, too. Hence, by using for example approximate sequences, $\|\xi_0\|_{E''} = 1$.

Furthermore, $\xi_0(f_0) = d_0 > 1$ and for every $f \in F$, we have $\xi_0(f) = 0$. By a Hahn-Banach standard type result this functional ξ_0 can be extended with the same norm, to the whole space, E'.

So, we have just proved the existence of a functional

$$\xi \in E'',\ \|\xi\|_{E''} = 1 \,\text{and}\, \xi(f_0) > 1,\ \xi(f) = 0,\ \text{for every } f \in F.$$

Let us denote W to be the set

$$W = \left\{ x \in B_E,\ |f_0(x)| < \frac{1}{2} \right\}.$$

Since f_0 is continuous, it is continuous in origin and so there exists $\delta > 0$ such that for every x with $\|x\| < \delta$, $|f_0(x)| < \frac{1}{2}$. It follows that W is a neighborhood of the origin. Since $V_n \subseteq U_n$, $(\forall)\ n \in \mathrm{N}^*$ and $(U_n)_{n \in \mathrm{N}^*}$ is strictly nonincreasing to $\{0\}$, it follows that there exists an integer number $n_0 \in \mathrm{N}^*$, such that

$$V_{n_0} \subseteq W. \tag{3}$$

Now, it is clearly that $x \in V_{n_0}$, and so

$$|f(x) - \xi(f)| < \varepsilon_{n_0},\ \text{for every } f \in \Phi_{n_0}.$$

Consider now

$$G_1 := \left\{ \eta \in E'',\ |(\eta - \xi)(f_0)| < \frac{1}{2} \right\}$$

and

$$G_2 := \left\{ \eta \in E'',\ |(\eta - \xi)(f_0)| < \frac{1}{2} \right\}.$$

such that G_1 and G_2 are neighborhoods of $\xi \in E''$, in the weak topology $\sigma(E'', E')$. Therefore, $G \overset{\text{not}}{:=} G_1 \cap G_2 \in \mathcal{V}_{\sigma(E'',E')}(\xi)$.

But, from Goldstine's lemma ([1], Lemma III.4),

$$\overline{J(B_E)} = B_{E''},$$

where the adherence is taken in the weak topology $\sigma(E'', E')$, and $J : E \to E''$ is the canonical application $(Jx)(f) := f(x)$, $(\forall)\ x \in E$, $f \in E'$. We deduce that for $\xi \in B_{E''}$

$$G \cap J(B_E) \neq \emptyset,$$

which means the existence of an element $x_1 \in B_E$ such that

$(J(x_1) \in G)$ if and only if $(J(x_1) \in G_1$ and $J(x_1) \in G_2)$.

Hence,

$$|(J(x_1) - \xi)(f_0)| < \frac{1}{2}$$

and

$$|(J(x_1) - \xi)(f)| < \varepsilon_{n_0},$$

for every $f \in \Phi_{n_0}$. Then

$$|f_0(x_1) - \xi(f_0)| < \frac{1}{2} \tag{4}$$

and

$$|f(x_1) - \xi(f)| < \varepsilon_{n_0}, \tag{5}$$

for every $f \in \Phi_{n_0}$. Therefore by (5) we deduce that $x_1 \in V_{n_0}$ and by (3), $x_1 \in W$. This allows us to conclude that

$$|f_0(x_1)| < \frac{1}{2}. \tag{6}$$

By other part, by (4), one gets

$$\frac{1}{2} < \xi\left(f_0\right) - \frac{1}{2} < f_0\left(x_1\right) < \xi\left(f_0\right) + \frac{1}{2},$$

and hence

$$f_0\left(x_1\right) > \frac{1}{2}. \tag{7}$$

Since the relations (6) and (7) are contradictory, our supposing $\overline{F} \neq E'$ is not true. Therefore, $\overline{F} = E'$.

For the end of the proof, let us suppose $\overline{D} \neq E'$. It follows that there exists $h_0 \in E' \backslash \overline{D}$. Through a classical functional reason, suggested by the Hahn-Banach theorem, we obtain a closed hyperplane $[\Psi = \alpha]$, $\alpha \in \mathbb{R}$ which separates strictly the sets $\{h_0\}$ and $\overline{D}$. We have

$$\Psi\left(f\right) < \alpha < \Psi\left(f h_0\right), \ (\forall)\ f \in D.$$

Hence, $\Psi\left(f\right) = 0$, $(\forall)\ f \in D$ and $\alpha > 0$. So, by considering $g \in D$ arbitrary, (a linear combination of the elements of D), we get $\Psi\left(g\right) = 0$. It results immediately that

$$\Psi\left(g\right) = 0, \text{ for every } g \in \overline{\mathrm{Space}\left(D\right)} = \overline{F} = E'.$$

This enables us to obtain $\Psi \equiv 0$; but this fact which is in contradiction with $\alpha > 0$. Therefore, we have just found that the set D counterable and dense in E' and then the space E' is separable. The proof is now complete.

3 Remark

We can draw up an example where we prove the existence of sequences $(f_n)_{n\in\mathbb{N}^*} \subset E'$, where E is a real Banach space, having the property $\|f_n\|_{E'} = 1$, for every $n \in\mathbb{N}^*$ which do not contain any weak $\sigma(E', E)$ convergent subsequences.

Indeed, let us consider the space $E := l^\infty$ (the space of real bounded sequences) and the sequence defined by

$$f_n := (0, 0, ..., 0, 1, 0, ...)\,, \text{ for every } n \in \mathbb{N}^*, \tag{8}$$

where 1 is on the n-th position. Obviously, we have

$$\|f_n\|_{E'} = 1, \text{ for every } n \in \mathbb{N}^*.$$

If, by contradiction, $(f_n)_{n\in\mathbb{N}^*}$ would contain a weak $\sigma(E', E)$ convergent subsequence then for every $\xi \in l^\infty$, the sequences $(f_{k_n})_{n\in\mathbb{N}^*}$ would be convergent in $\mathbb{R}$.

But, for

$$\xi \overset{\text{def}}{:=} (0, ..., -1, 0, ..., 1, 0, ..., 0, (-1)^n\,, 0, ...) \in l^\infty, \tag{9}$$

where $(-1)^n$ is on the n-th position, we have, by (8) and (9) $\sum_{i=1}^{\infty} (f_{k_n})_i \cdot \xi_i = \xi_{k_n} = (-1)^n$, for every $n \in \mathbb{N}^*$, and therefore the sequence $(-1)^n$ is not a convergent sequence in $\mathbb{R}$.

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