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Duality for multiobjective stochastic programming

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Abstract

This paper presents two ways of constructing the dual of the vectorial stochastic programming problem with simple recourse in an original way. The first method used is the transformation of vectorial stochastic programming problem in a stochastic program with a single objective function to which a dual in the sense of Wolfe is constructed. The second method is the dual's construction after the initial problem was transformed, in turn, into a deterministic vectorial programming problem and then into one with a single objective function.

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1 Introduction

The study of certain aspects concerning the dual of the vectorial stochastic programming problem with a single objective function was the preoccupation of certain authors who elaborated a theory of duality for different types of the problems. Among those who obtained fundamental results in this sense, we may mention Wilson [5], Ziemba [7] and Rockafellar and Wets [3], who created the basis of duality in stochastic programming. For the vectorial stochastic programming problem these aspects haven't been but little analyzed and we intend to approach the possibility of constructing the dual problem of vectorial stochastic programming with a simple recourse problem. We consider the vectorial stochastic problem with the simple recourse.

PVR

$$V max \;\; Z(x) \tag{1}$$

subject to:

$$Tx + y = \xi(\omega)$$

$$x \geq 0, \ y \geq 0$$

where $Z : \mathbb{R}^n \to \mathbb{R}^r$, $Z = (z_1, z_2, ..., z_r)$ is a vectorial function, $z_k(x, \xi) = g_k(x) + E_\xi\{Q_k(x, \xi)\}, (k = 1, 2, ..., r), g_k$ is a linear function, $\xi = (\xi_1, \xi_2, ..., \xi_m)^T$ is a random vector, defined on the probability space $\{\Omega, \mathcal{K}, \mathcal{P}\}, \omega \in \Omega$, T is a m x n matrix, x is a $n-vector$, y is a $m-vector$, E_ξ denotes the mean operator,

$$Q_k(x, \xi) = \min_{y \in Y_\xi} q_k y, (k = 1, 2, ..., r),$$

$$Y_\xi = \{y \in \mathbb{R}^m | y \leq 0, Tx + y = \xi(\omega)\}$$

where q_k is the kth row of the penalties matrix $Q = (q_{kj}), k = 1, 2, ..., r$; $j = 1, 2, ..., m$. The function $Q_k(x, \xi)$ is called *recourse function*.

The **PVR** model has the following interpretation: if, for a given decision x and realization $\xi(\omega)$, the constraint $Tx = \xi(\omega)$ is violated, we could provide a recourse decision y, such as to compensate its constraint's violation by satisfying $Tx + y = \xi(\omega)$. This extra effort is assumed to cause a penalty of $q_k(k = 1, 2, ..., r)$ per unit for the kth objective function.

We denote by:

$Q_k(\chi) = E_\xi\{Q_k(x, \xi)\} = E_\xi\{h_k(\xi(\omega) - \chi)\}, (k = 1, 2, ..., r)$, where $y = \xi(\omega) - \chi$.

The problem (1) becomes:

$$V max \ (g_1(x) - Q_1(\chi), \ g_2(x) - Q_2(\chi), ..., \ g_r(x) - Q_r(\chi)) \tag{2}$$

subject to:

$$Tx + y = \xi(\omega)$$

$$x \geq 0, \quad y \geq 0$$

We consider two possible ways of constructing the dual of (2) problem:

a. We change the vectorial stochastic programming problem into a stochastic problem with a single objective whose dual we construct.

b. We change the vectorial stochastic programming problem in its vectorial deterministic equivalent and we construct its dual using a result obtained by Kolumbán [2].

2 The duality in stochastic programming problem with simple recourse

We approach the first way of constructing the (1) problem's dual which we will change into a stochastic program with a single objective function using weights of the objectives denoted by α_k, $(k = 1, 2, ..., r)$. The problem (2) is equivalent with the following stochastic programming problem with a single objective function:

$$maxZ^*(x,\chi) = \sum_{k=1}^{r} \alpha_k z_k(x,\chi), (k = 1, 2, ..., r) \tag{3}$$

subject to:

$$Tx - y = \xi$$

$$x \geq 0, \quad y \geq 0$$

where the weights fulfill the conditions: $\sum_{k=1}^{r} \alpha_k = 1$, $\alpha_k \in [0, 1]$, $k = 1, 2, ..., r$.

In the case in which the functions z_k, $(k = 1, 2, ..., r)$ can not be summed they change into utility functions in Neumann-Morgenstern sense and we obtain $z_k'(x,\chi) = u_k z_k(x,\chi) + v_k$, $(k = 1, 2, ..., r)$ where u_k and v_k are determinated from the system:

$$\begin{cases} u_k m_k + v_k = 1 \\ u_k M_k + v_k = 0, (k = 1, 2, ..., r) \end{cases}$$

where m_k and M_k are the minimum and maximum of $z_k(x,\chi)$ $(k = 1, 2, ..., r)$ on the domain of the possible solutions, determining the synthesis function $Z^*(x,\chi) = \sum_{k=1}^{r} \alpha_k z_k'(x,\chi)$.

It should be observed that $Z^*(x,\chi) = \sum_{k=1}^{r} \alpha_k [g_k(x) - Q_k(\chi)]$ is a convex function being a sum of convex functions and the set of constraints is convex being an intersection of convex regions.

We write the problem(3) in the form:

$$maxZ^*(x,\chi) = g^*(x) - Q^*(\chi), \quad (k = 1, 2, ..., r) \tag{4}$$

subject to:

$$Tx - y = \xi(\omega)$$
$$x \leq 0, \quad y \leq 0$$

where we noted

$$g^*(x) - Q^*(\chi) = a_1g_1(x) + a_2g_2(x) + \ldots + a_rg_r(x) - (a_1Q_1(\chi) + a_2Q_2(\chi) + \ldots + a_rQ_r(\chi))$$

knowing that:

$$Q_k(\chi) = E_\xi\{h_k(\xi - \chi)\}, \quad (k = 1, 2, \ldots, r)$$

We can write that

$$Q^*(\chi) = E_\xi\{h^*(\xi - \chi) = E_\xi\{(a_1h_1(\xi - \chi) + a_2h_2(\xi - \chi) + \ldots + a_rh_r(\xi - \chi))\}.$$

Bringing the problem (1) to this form we may use a result belonging to Ziemba [7].

Theorem 1 *Assume that* $h_k, (k = 1, 2, \ldots, r)$ *defined on* $\mathbb{R}^m$ *is integrable and continuously differentiable (except possibly on the set of measure zero), and there exists* m *integrable function* $G_i, i = 1, 2, \ldots, m$ *such that*

$$\left[\frac{\partial h^*(\xi - \chi)}{\partial \chi_i}\right] \leq G_i^*(\xi), \quad i = 1, 2, \ldots, m.$$

Then $Q^*(\chi) = E_\xi\left[h^*(\xi - \chi)\right]$ *is continuously differentiable (except possibly on the set of measure zero) and*

$$\frac{\partial Q^*(\chi)}{\partial \chi_i} = E_\xi\left[\frac{\partial h^*(\xi - \chi)}{\partial \chi_i}\right] \quad , i = 1, 2, \ldots, m.$$

If we assume that Z^* is differentiable then using the Wolfe formulation [6], a dual of problem (1) is:

$$\text{(5)} \qquad minW(x, \chi, v) = g^*(x) - Q^*(\chi) + v^T(Tx - x)$$

subject to:

$$\nabla_x W(x, \chi, v) = \nabla_x g^*(x) + T^T v = 0$$
$$\nabla_x W(x, \chi, v) = -\nabla_x Q^*(\chi) - v = 0, \quad v \leq 0.$$

Under these conditions Wolfe [6] has proved the following duality theorem which we present without proof.

Theorem 2 *Suppose that Z^* is differentiable and convex on the open convex $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ and assume that the constraints of the problem (4) are fulfilled. If $(\overline{x}, \overline{\chi})$ is a solution of problem (4) there exists $\overline{v}$ such that $(\overline{x}, \overline{\chi}, \overline{v})$ is a solution to (5) and the extremes of the two problems are equal.*

We present now the second way of transforming the vectorial stochastic programming problem into a deterministic vectorial programming problem to which we can construct a dual using a result that is due to *Kolumbán*[2]. For finding the deterministic equivalent of the problem we consider the two stages of the problem with simple recourse.

The *second stage* is as follows:

$$V\min(q_1 y, q_2 y, \ldots, q_r y) \tag{6}$$

subject to:

$$Tx + y = \xi(\omega)$$
$$y \leq 0$$

Let $Q_k(x, \xi)$ be the optimum of the problem (6) and we will note $Q_k(x) = E_\xi\{Q_k(x, \xi)\}$ $(k \in I)$.

The *first stage* of the problem (1) is:

$$V\max(g_1(x) - Q_1(x), g_2(x) - Q_2(x), \ldots, g_r(x) - Q_r(x)) \tag{7}$$

subject to:

$$x \leq 0, \; x \in K_1$$

where we note $K_1 = \{x \in D|$ for every $s \in S$ there exists $y \leq 0$ such that $Tx + y = s\}$, $D = \bigcap_{k=1}^{r} D_k$, $\quad D_k = \{x \in \mathbb{R}^n | Q_k(x, \xi) < +\infty$ almost surely$\}$, $S \in \mathbb{R}^m$ being the support of the distribution function of the random variable ξ $(P(\xi \in S) = 1)$.

Taking into account that in (6) there is the condition that $y \leq 0$ if we note s_0 the lower bound of S, the set K_1 is $K_1 = \{x \in \mathbb{R}^n | Tx \geq s_0\}$ and we note $K = \mathbb{R}^n_+ \bigcap K_1$. Under these conditions we shall try to find the deterministic equivalent of the problem (1).

In [1] was demonstrated that the following theorem holds:

Theorem 3 *$x^0 \in K$ is an efficient solution for (1) if and only if x^0 is an efficient solution for the following multiobjectiv linear programming problem:*

$$\max(c_1 x + q_1 T x, c_2 x + q_2 T x, \ldots, c_r x + q_r T x)$$

subject to:

$$x \in K$$

Therefore the deterministic equivalent of (1) is the following vectorial programming problem

$$\text{(8)} \qquad V\max(c^1x + d^1Tx, c^2x + d^2Tx, \ldots, c^rx + d^rTx)$$

subject to:

$$Tx \leq s^0$$

$$x \geq 0$$

where $s^0 = (b^1, b^2, \ldots, b^m)$.

Let d^kx be the linear functions, where $d^k = c^k + d^kT \quad (k = 1, 2, \ldots, r)$. The deterministic equivalent of the problem (1) is as follows:

$$\text{(9)} \qquad V\max(d^1x, d^2x, \ldots, d^rx)$$

subject to:

$$Tx \leq s^0$$

$$x \geq 0$$

or:

$$\text{(10)} \quad V\max(d^{11}x^1 + d^{12}x^2 + \cdots + d^{1n}x^n, \ldots, d^{r1}x^1 + d^{r2}x^2 + \cdots + d^{rn}x^n)$$

subject to:

$$\text{(11)} \qquad \sum_{j=1}^{n} t^{ij}x^j \leq b^i, \qquad i = 1, 2, \ldots, m$$

$$x^j \geq 0, \qquad j = 1, 2, \ldots, n.$$

The vectorial programming problem is transformed into a problem with a single objective function using the real numbers $y^{m+1}, y^{m+2}, \ldots, y^{m+r}$ where $y^{m+k} \geq 0 \quad (k = 1, 2, \ldots, r)$ and $\sum_{k=1}^{r} y^{m+k} \geq 0$.

We get that (10)-(11) is:

$$\text{(12)} \qquad \max \sum_{k=1}^{r} \left(\sum_{j=1}^{n} d^{kj}x^j \right) y^{m+k}$$

subject to:

$$\sum_{j=1}^{n} t_{ij} \quad x_j \leq b_i \qquad i = 1, 2, ..., m \tag{13}$$

$$x_j \geq 0 \qquad j = 1, 2, ..., n.$$

We show using a result due to *Kolumbán*[2] that a dual of (12)–(13) is as follows:

$$min \sum_{i=1}^{m} b_i y_i \tag{14}$$

subject to:

$$\sum_{i=1}^{m} t_{ij} y_i \geq \sum_{k=1}^{r} d_{kj} y_{m+k}, \qquad j = 1, 2, ..., n \tag{15}$$

$$\sum_{k=1}^{r} y_{m+k} \geq 0, \quad y_{m+k} \geq 0, \quad k = 1, 2, ..., r.$$

Let X and X′ be nonempty convex sets. Let M be the set of maximal elements of X and M the set of the optimal elements of X′.

Theorem 4 • *1 If the system (13) or the system (15) is incompatible then M and M′ are nonempty.*

- *2 If both (13) and (15) systems are compatible then M and M′ are nonempty.*

- *3 The element $(x_1, x_2, ..., x_n)$ which satisfies (13) is contained in M if and only if there exists an element $(y_1, y_2, ..., y_{m+r})$ contained in M′ such that:*

$$\sum_{i=1}^{m} b_i y_i = \sum_{k=1}^{r} \left(\sum_{j=1}^{n} d_{kj} x_j \right) y_{m+k} \tag{16}$$

- *4 The element $(y_1, y_2, ..., y_{m+r})$ which satisfies (15) is contained in M′ if and only if there exists an element $(x_1, x_2, ..., x_n)$ contained in M such that the (16) equality hold.*

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