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Numerical solution of two-dimensional nonlinear Fredholm integral equations of the second kind by spline functions

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Abstract

In this paper we shall investigate the numerical solution of two-dimensional Fredholm integral equation by Galerkin method using as approximating subspace a special space of spline functions. The estimation of the error as well as the convergence of the given procedures are studied. Some numerical examples illustrate the efficiency of the method.

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1 Introduction

The integral equations provide an important tool for modeling a numerous phenomena and processes and also for solving boundary value problems for both ordinary and partial differential equations. Their historical development is closely related to the solution of boundary value problems in potential theory. Progress in the theory of integral equations also had a great impact on the development of functional analysis. Reciprocally, the main results of the theory of compact operators have taken the leading part to the foundation of the existence theory for integral equations of the second kind. In the last decades there has been much interest in numerical solutions of integral equations. The Nystrom method and the collocation method are, probably, the two most important approaches for the numerical solution of these integral equations. But also many other methods are known for the approximate solution of the integral equations. For a comprehensive study of both the theory and the numerical solution of integral equations we refer to monographs of Hackbusch [7], Athinson [2] and Baker [4]. Recently, very important results, containing the Galerkin and iterated Galerkin methods, respectively the iterated collocation method for linear Fredholm integral equations have been published by Chen and Xu [5] and Lin, Sloan and Xie [11].Fewer numerical methods are known for the nonlinear integral equations and especially for several-dimensional Fredholm integral equations. In this paper we will be concerned to the Galerkin and iterated Galerkin methods for the two-dimensional nonlinear Fredholm integral equations of the second kind, using as approximating subspace a special spline function space. Such methods using the Richardson extrapolation of Galerkin solutions have been investigated by Han and Wang [9].

Let consider the following nonlinear two-dimensional Fredholm integral equations of the second kind

$$(1) \qquad u\left(x,y\right) = \int\limits_a^b \int\limits_c^d K\left(x,y,t,s,u\left(t,s\right)\right) dtds + f\left(x,y\right), \left(x,y\right) \in D := [a,b] \times [c,d]$$

where $K : D \times D \times \Re \to \Re$ is a continuous nonlinear in u given function, $f : D \to \Re$ is also continuous given function and the two-variable function u is the unknown function.

Introducing the Uryson integral operator defined by:

$$\left(Ku\right)\left(x,y\right) := \int\limits_a^b \int\limits_c^d K\left(x,y,t,s,u\left(t,s\right)\right) ds$$

the equation (1) takes the operator form

$$(2) \qquad u = Ku + f$$

The most used numerical method for (1) are the collocation and Galerkin methods, as we can see in [1]-[3], [6], [11]-[14].

In [8], [13] a general theory for solving numerically linear and nonlinear one-dimensional Fredholm integral equations are given and the error analysis are also investigated. Also the error expansions for the numerical solution of one-dimensional linear integral equations have been discussed by Marchuck and Shaidurov [12] and Baker [4]. McLean [13] and Lin et al. [11] obtained the asymptotic error expansion for numerical solution of Fredholm integral equations, including the Nystrom method, iterated collocation method and iterated Galerkin method. In this paper, following the idea of Han and Wong [9] we shall consider the two-dimensional equation (1) by using the two-dimensional polynomial spline functions of degree (p,q) and interpolatory quadrature formulas to evaluate the integrals occurring in the Galerkin and iterated Galerkin methods. If the step-sizes are denoted by h and k, the error estimation will be obtained with terms in h^{2p} and k^{2q}.

Throughout in this paper we assume that the following conditions are satisfied:

i Equation (1) has an unique solution $u \in C^{r+1}(D)$ for a given $r \in \aleph$;

ii $(I - Ku)$ is nonsingular for the solution u;

iii Functions K and f are smooth enough.

2 The Spline-Galerkin method

Let $\Delta^{(1)}$ and $\Delta^{(2)}$ denote, respectively the uniform partitions of $[a,b]$ and $[c,d]$:

$$\Delta^{(1)} : a = x_0 < x_1 < ... < x_M = b, \Delta^{(2)} : c = y_0 < y_1 < ... < y_N = d$$

with:

$$h := (x_{i+1} - x_i) = \frac{b-a}{M}; k := (y_{j+1} - y_j) = \frac{d-c}{N}.$$

These partitions define a grid for D denoted by:

$$\Delta_{M,N} := \Delta^{(1)} \times \Delta^{(2)} = \{(x_m, y_n) : 0 \leq m \leq M; 0 \leq n \leq N\}.$$

Set

$$I_0^{(1)} := [x_0, x_1], I_m^{(1)} :=]x_m, x_{m+1}], m = 1, 2, ..., M-1;$$
$$I_0^{(2)} := [y_0, y_1], I_n^{(2)} :=]y_n, y_{n+1}], n = 1, 2, ..., N-1.$$

and let $I_{m,n}$ be the two-dimensional rectangles defined by

$$I_{m,n} := I_m^{(1)} \times I_n^{(2)}; m = 0, 1, ..., M-1;$$

We shall use the following polynomial spline functions finite element space:

$$S_{p,q}^{(-1)}(\Delta_{M,N}) := \left\{ v : v\mid_{I_{m,n}} =: u_{m,n} \in \mathcal{P}_{p,q}, 0 \leq m \leq M-1; 0 \leq n \leq N-1 \right\}$$

where $\mathcal{P}_{p,q}$ denotes the space of real polynomials of degree p in x and degree q in y. For simplicity, we shall write this spline subspace by $S_{p,q}^{(-1)}$. The superscript (-1) in the notation of spline finite element space emphasize that spline spaces $S_{p,q}^{(-1)}$ is not a subspace of $C(D)$, i.e. the segments of splines are not continuous connected.

Now, the spline Galerkin method is the following:

Find $u^{hk} \in S_{p-1,q-1}^{(-1)}$ such that

$$\left(u^{hk}, v\right) = \left(Ku^{hk}, v\right) + (f, v), \forall v \in S_{p-1,q-1}^{(-1)} \tag{3}$$

where $(\bullet, \bullet)$ denotes the usual inner product in $L_2(D)$.

If P denotes the orthogonal projection of $L_2(D)$ onto $S^{(-1)}_{p-1,q-1}$, then the spline Galerkin method (3) can be equivalently rewritten: Find $u^{hk} \in S^{(-1)}_{p-1,q-1}$ such that

$$u^{hk} = PKu^{hk} + Pf. \tag{4}$$

The iterated Galerkin spline solution, $\overline{u}^{hk}$, corresponding to the above Galerkin spline solution u^{hk} is given by:

$$\overline{u}^{hk}(x,y) = \left(Ku^{hk}\right)(x,y) + f(x,y), (x,y) \in D. \tag{5}$$

For the iterated Galerkin solution $\overline{u}^{hk}$ it is easy to show that:

$$(I - KP)\overline{u}^{hk} = f \tag{6}$$

and that

$$P\overline{u}^{hk} = u^{hk}. \tag{7}$$

To give an explicit formula for Pu we denote the inner product in the real Hilbert space $L_2[0,1]$ as usual by

$$(u,v) = \int_0^1 u(t)\, v(t)\, dt.$$

Let $\varphi_0, \varphi_1, \varphi_2, ...$ be the sequence of orthogonal polynomials associated with the above inner product, i.e. φ_i is a polynomial of degree i and

$$(\varphi_i, \varphi_j) = \delta_{i,j}, i, j \geq 0.$$

Let $L_0(t) = 1$ and

$$L_i(t) := \frac{1}{2^i L!} \frac{d^i}{dt^i} \left(t^2 - 1\right)^i, i \geq 1$$

the Legendre polynomials of degree i. Then the orthogonal polynomial φ_i are related to the Legendre polynomials L_i by

$$\varphi_i(t) := \sqrt{2i+1} L_i(2t-1).$$

Now set $\Psi_j(s) := \sqrt{2j+1} L_j(2s-1)$.

Defining the piecewise functions

$$\varphi_{im}(x) := \begin{cases} \frac{1}{\sqrt{h}}\varphi_i\left(\frac{x-x_m}{h}\right), x \in [x_m, x_{m+1}] \\ 0, x \in [a,b] \setminus [x_m, x_{m+1}] \end{cases}$$
$$\Psi_{jn}(y) := \begin{cases} \frac{1}{\sqrt{k}}\Psi_j\left(\frac{y-y_n}{k}\right), y \in [y_n, y_{n+1}] \\ 0, x \in [c,d] \setminus [y_n, y_{n+1}] \end{cases}$$

then the functions

$$\{\varphi_{im}(x)\,\Psi_{jn}(y)\}\ (0 \le i \le p-1, 0 \le m \le M-1, 0 \le j \le q-1$$

and $0 \le n \le N-1$) form an orthogonal basis of the spline space $S^{(-1)}_{p-1,q-1}$.

Therefore

$$(Pu)(x,y) = \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(\varphi_{im}\Psi_{jn}, u)\,\varphi_{im}(x)\,\Psi_{jn}(y)\,. \tag{8}$$

The solution processes for equation (3) leads to an algebraic nonlinear system in which each coefficient of the system is a definite integral. Because the integrals occurring in (3) and (5) cannot be obtained in general exactly, these integrals have to be approximated by suitable quadrature formulas. When the quadrature formulas are given, the method is called the discrete spline Galerkin method. We shall introduce such a discrete method.

Let $c_1, c_2, ..., c_{p-1}$ be the Gauss knots in the interval $]0,1[$.

The following Gauss quadrature formula

$$\int_0^1 g(t)dt \approx \sum_{i=0}^{p-1} w_i g(c_i) =: R(g) \tag{9}$$

with $0 < c_0 < c_1 < ... < c_{p-1} < 1$ is an interpolatory quadrature rule which is exact for all polynomials of degree $2p-1$, but not exact for any polynomials of degree $2p$ or higher.

Let $x_{m,i} := x_m + c_i h$ $(m = 0,1,...,M-1; i = 0,1,...,p-1)$.

From (9) we obtain the following composite quadrature rule:

$$\int_a^b g(t)dt \approx h\sum_{m=0}^{M-1}\sum_{i=0}^{p-1} w_i g(x_{i,m}) =: R_h(g)\,. \tag{10}$$

Similarly, if d_j, $j = 0,1,...q-1$ are the Gauss points in $]0,1[$, then the interpolatory quadrature rule

$$\int_0^1 g(t)dt \approx \sum_{j=0}^{q-1} \overline{w}_j g(d_j) =: S(g)$$

furnishes the composite quadrature rule

$$\int_c^d g(t)dt \approx k \sum_{n=0}^{N-1} \sum_{j=0}^{q-1} \overline{w}_j g\left(y_{n,j}\right) =: S_k\left(g\right) \tag{11}$$

where $y_{n,j} := y_n + d_j k$, $n = 0, 1, ..., N-1$, $j = 0, 1, ..., q-1$.

We define a discrete integral operator K_{hk} by

$$\left(K_{hk}u\right)(x,y) := hk \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} w_i \overline{w}_j K\left(x, y, x_{m,i}, y_{n,j}, u\left(x_{m,i}, y_{n,j}\right)\right). \tag{12}$$

Using (10) and (11) we define a discrete semidefinite inner product:

$$\left(f, g\right)_{hk} := hk \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} w_i \overline{w}_j f\left(x_{m,i}, y_{n,j}\right) g\left(x_{m,i}, y_{n,j}\right), f, g \in C\left(D\right) \tag{13}$$

We introduce now a discrete analog of the orthogonal projection operator P, denoted by Q and defined as follows:

For $u \in C\left(D\right)$, define $z := Qu$ to be the unique element in $S_{p-1,q-1}^{(-1)}$ that satisfies:

$$\left(z, \Phi\right)_{hk} = \left(u, \Phi\right)_{hk}, \tag{14}$$

It is clear that $Q : C\left(D\right) \to S_{p-1,q-1}^{(-1)}$ is a projection operator.

By effective calculating of Qu we obtain:

$$\left(Qu\right)(x,y) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(\varphi_{im}\Psi_{jn}, u\right) \varphi_{im}\left(x\right) \Psi_{jn}\left(y\right). \tag{15}$$

Using now the projection operator Q, the discrete Galerkin method for solving the equation (2) is defined as follows:

Find $z^{hk} \in S_{p-1,q-1}^{(-1)}$ such that

$$\left(I - QK_{hk}\right) z^{hk} = Qf. \tag{16}$$

The iterated discrete spline Galerkin solution $\overline{z}^{hk}$, corresponding to discrete spline Galerkin solution z^{hk} is given by

$$\overline{z}^{hk}\left(x,y\right) = \left(K_{hk}z^{hk}\right)(x,y) + f\left(x,y\right), (x,y) \in D. \tag{17}$$

For the iterated discrete spline Galerkin solution of (17) we have

$$Q\overline{z}^{hk} = \left(QK_{hk}z^{hk} + Qg\right) = z^{hk}.$$

Substituting it back in (17) we obtain that $\overline{z}^{hk}$ satisfies

$$(I - QK_{hk})\,\overline{z}^{hk} = f. \tag{18}$$

As a spline approximating solution of the problem (2) we shall consider the iterated spline Galerkin solution $\overline{z}^{hk}$.

3 The estimation of the error

First we need the asymptotic error expansion of the discrete orthogonal projection Qu.

Theorem 1. Let $r \geq \max(p,q)$ be an integer and let $u \in C^{r+1}(D)$. Then, for any $(x,y) \in \,]x_m, x_{m+1}[\, \times \,]y_n, y_{n+1}[$, $m = 0,1,...,M-1$, $n = 0,1,...,N-1$ we have:

$$\begin{aligned} Qu\,(x,y) = \sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{r-\mu} h^{\mu}k^{\upsilon}u^{(\mu,\upsilon)}(x,y)\,\Phi_{\mu}\left(\tfrac{x-x_m}{h}\right)\Psi_{\upsilon}\left(\tfrac{y-y_n}{k}\right) + \\ +O\left(h^{r+1}+k^{r+1}\right) \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Phi_{\mu}(\tau) &:= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{p-1}\varphi_i(c_\alpha)\,\varphi_i(\tau)\,\tfrac{(c_\alpha-\tau)^{\mu}}{\mu!}\text{and} \\ \Psi_{\upsilon}(\tau) &:= \sum_{\beta=0}^{q-1}\sum_{j=0}^{q-1}\Psi_j(d_\beta)\,\Psi_j(\theta)\,\tfrac{(d_\beta-\theta)^{\mu}}{\upsilon!}. \end{aligned}$$

Proof. Let $(x,y) \in \,]x_m, x_{m+1}[\, \times \,]y_n, y_{n+1}[$. From (13) and recalling the definitions of φ_{im} and Ψ_{jn} we have:

$$(u, \varphi_{im}\Psi_{jn})_{hk} = \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{q-1} w_\alpha \overline{w}_\beta \varphi_i(c_\alpha)\,\Psi_j(d_\beta)\,u\left(x_m + c_\alpha h, y_n + d_\beta k\right).$$

Let $x = x_m + \tau h$ and $y = y_n + \theta k$, then , using Taylor‘s theorem and writing it as polynomials in h and k we obtain:

$$\begin{aligned} (u, \varphi_{im}\Psi_{jn})_{hk} &= \sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{q-1} w_\alpha \overline{w}_\beta \varphi_i(c_\alpha)\,\Psi_j(d_\beta)\,u\left(x + (c_\alpha-\tau)\,h, y + (d_\beta-\theta)\,k\right) = \\ &= \sum_{\mu=0}^{r}\sum_{\upsilon=0}^{r-\mu} h^{\mu}k^{\upsilon}u^{(\mu,\upsilon)}(x,y)\left(\sum_{\alpha=0}^{p-1} w_\alpha\varphi_i(c_\alpha)\,\tfrac{(c_\alpha-\tau)^{\mu}}{\mu!}\right)\cdot \\ &\quad\cdot\left(\sum_{\beta=0}^{q-1}\overline{w}_\beta\Psi_j(d_\beta)\,\frac{(d_\beta-\theta)^{\mu}}{\upsilon!}\right) + O\left(h^{r+1}+k^{r+1}\right). \end{aligned}$$

Substituting the above expression into (15) we have:

$$(Qu)(x,y) = \sum_{\mu=0}^{r}\sum_{\upsilon=0}^{r-\mu} h^{\mu}k^{\upsilon}u^{(\mu,\upsilon)}(x,y)\left(\sum_{\alpha=0}^{p-1}\sum_{i=0}^{p-1}\varphi_i(\alpha)\varphi_i(\tau)\frac{(c_\alpha-\tau)^{\mu}}{\mu!}\right)\cdot$$
$$\cdot\left(\sum_{\beta=0}^{q-1}\sum_{j=0}^{q-1}\Psi_j(d_\beta)\Psi_j(\theta)\frac{(d_\beta-\theta)^{\mu}}{\upsilon!}\right) =$$
$$= \sum_{\mu=0}^{r}\sum_{\upsilon=0}^{r-\mu} h^{\mu}k^{\upsilon}u^{(\mu,\upsilon)}(x,y)\,\Phi_\mu\left(\tfrac{x-x_m}{h}\right)\Psi_\upsilon\left(\tfrac{y-y_n}{k}\right) + O\left(h^{r+1}+k^{r+1}\right)$$

and the theorem is proved.

Noting that c_i, $i=0,1,...,p-1$ are Gauss point in the interval $]0,1[$, the quadrature rule (9) is an interpolation quadrature rule and we have:

$$\begin{aligned}\Phi_\mu(\tau) &:= \sum_{\alpha=0}^{p-1}\sum_{i=0}^{p-1}\varphi_i(c_\alpha)\varphi_i(\tau)\frac{(c_\alpha-\tau)^{\mu}}{\mu!} = \\ &= \int_0^1\sum_{i=0}^{p-1}\varphi_i(\xi)\varphi_i(\tau)\frac{(\xi-\tau)^{\mu}}{\mu!}d\xi,\ \mu\le p.\end{aligned} \tag{20}$$

But using the Cristoffel-Darboux identity we have

$$\sum_{i=0}^{p-1}\varphi_i(\xi)\varphi_i(\tau) = \frac{a_{p-1}}{a_p}\cdot\frac{\varphi_p(\xi)\varphi_{p-1}(\tau)-\varphi_{p-1}(\xi)\varphi_p(\tau)}{\xi-\tau} \tag{21}$$

where a_p is the leading coefficient of the polynomial φ_p. Because $\varphi_0, \varphi_1, ...$ are orthogonal polynomials, it is easy to see that $\Phi_\mu(\tau)=0$ for $1\le\mu\le p-1$ and similarly $\Psi_\upsilon(\tau)=0$ for $1\le\upsilon\le q-1$.

From Theorem 1 we have the following corollary.

Corollary 1. Let $r\ge\max(p,q)$ be an integer and let $u\in C^{r+1}(D)$. Then, for any $(x,y)\in\,]x_m,x_{m+1}[\,\times\,]y_n,y_{n+1}[$, $m=0,1,...,M-1$, $n=0,1,...,N-1$ we have:

$$(Q-I)\,u(x,y) = \sum_{\mu=p}^{r-q} h^{\mu}u^{(\mu,0)}(x,y)\,\Phi_\mu\left(\tfrac{x-x_m}{h}\right) + \sum_{\upsilon=q}^{r} k^{\upsilon}u^{(0,\upsilon)}(x,y)\,\Psi_\upsilon\left(\tfrac{y-y_n}{k}\right) +$$
$$+\sum_{\mu=p}^{r-q}\sum_{\upsilon=q}^{r-\mu} h^{\mu}k^{\upsilon}u^{(\mu,\upsilon)}(x,y)\,\Phi_\mu\left(\tfrac{x-x_m}{h}\right)\Psi_\upsilon\left(\tfrac{y-y_n}{k}\right) + O\left(h^{r+1}+k^{r+1}\right)$$

where $\Phi_\mu(\tau)$ and $\Psi_\upsilon(\theta)$ are defined in Theorem 1.

Lemma 1.For $i=0,1,...,r$ and $j=0,1,...,r-1$ let $V_{i,j}\in C^{r+1-i-j}(D)$ and let be $V(x,y) := \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j V_{i,j}(x,y)$.

Then, for any $(x,y)\in\,]x_m,x_{m+1}[\,\times\,]y_n,y_{n+1}[$, $m=0,1,...,M-1$, $n=0,1,...,N-1$ holds:

$$QV(x,y) := \overline{V}_{0,0}(x,y) + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j\overline{V}_{i,j}\left(x,y,\frac{x-x_m}{h},\frac{y-y_n}{k}\right) + O\left(h^{r+1}+k^{r+1}\right)$$

where $\overline{V}_{0,0}(x,y,t,s) := 0$ for $i \neq 0$ and $j \neq 0$ and
$\overline{V}_{i,j}(x,y,t,s) := \sum\limits_{\mu=0}^{i}\sum\limits_{\upsilon=0}^{j} V^{(\mu,\upsilon)}(x,y)\,\Phi_\mu(t)\,\Psi_\upsilon(s)\,.$

Proof. From Theorem 1 we have for any $(x,y) \in \,]x_m, x_{m+1}[\, \times \,]y_n, y_{n+1}[$:

$$\begin{aligned} QV(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j QV_{i,j}(x,y) + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \cdot \\ \cdot \sum_{\mu=0}^{r-i-j}\sum_{\upsilon=0}^{r-i-j-\mu} h^\mu k^\upsilon V_{i,j}^{(\mu,\upsilon)}(x,y) \cdot \\ \cdot \Phi_\mu\left(\tfrac{x-x_m}{h}\right)\Psi_\upsilon\left(\tfrac{y-y_n}{k}\right) + O\left(h^{r+1}+k^{r+1}\right) \end{aligned} \tag{22}$$

and writing (22) as polynomials in h and k it follows:

$$\begin{aligned} QV(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \cdot \sum_{\mu=0}^{i}\sum_{\upsilon=0}^{j} V_{i-\mu,j-\upsilon}^{(\mu,\upsilon)}(x,y) \cdot \\ \cdot \Phi_\mu\left(\tfrac{x-x_m}{h}\right)\Psi_\upsilon\left(\tfrac{y-y_n}{k}\right) + O\left(h^{r+1}+k^{r+1}\right). \end{aligned}$$

Now, if $\overline{V}_{0,0}(x,y,t,s) := 0$ for $i \neq 0$ and $j \neq 0$ and
$\overline{V}_{i,j}(x,y,t,s) := \sum\limits_{\mu=0}^{i}\sum\limits_{\upsilon=0}^{j} V_{i-\mu,j-\upsilon}^{(\mu,\upsilon)}(x,y)\,\Phi_\mu(t)\,\Psi_\upsilon(s)$
we obtain Lemma 1.

Lemma 2. (Euler-McLaurin summation formula). Let $f \in C^{r+1}(D)$ and τ, θ with $0 \leq \tau \leq 1$, $0 \leq \theta \leq 1$. Then

$$\begin{aligned} hk \sum_{\mu=0}^{M-1}\sum_{\upsilon=0}^{N-1} f\left(x_\mu + \tau h, y_\upsilon + \theta k\right) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} \tfrac{h^i k^j}{i!j!} B_i(\tau)\, B_j(\theta) \cdot \\ \cdot \left[f^{(i-1,j-1)}(x,y)\right]_{x=a,y=c}^{b\ \ d} + O\left(h^{r+1}+k^{r+1}\right) \end{aligned}$$

where B_j are Bernoulli polynomials and

$$\begin{aligned} \left[f^{(-1,-1)}(x,y)\right]_{x=a,y=c}^{b\ \ d} := \int_a^b\int_c^d f(x,y)\,dxdy, \\ \left[f(x,y)\right]_{x=a,y=c}^{b\ \ d} := f(b,d) - f(b,c) - f(a,d) + f(a,c)\,. \end{aligned}$$

Lemma 3. Let $f \in C^{r+1}(D)$. Then we have the following cubature formula:

$$\begin{aligned} R_h S_k(f) = hk \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} w_i \overline{w}_j f\left(x_{m,i}, y_{n,j}\right) = \\ = \sum_{i=0}^{r}\sum_{j=0}^{r-i} \tfrac{h^i k^j}{i!j!} R(B_i)\, S(B_j) \cdot \left[f^{(i-1,j-1)}(x,y)\right]_{x=a,y=c}^{b\ \ d} + O\left(h^{r+1}+k^{r+1}\right). \end{aligned}$$

Now, let come to discuss the error expansing problem. We first consider linear two-dimensional Fredholm integral equation of the second kind:

$$u(x,y)=\int_a^b\int_c^d K(x,y,t,s,)\,u(t,s)\,dtds+f(x,y),(x,y)\in D \tag{23}$$

which may be written in the operator form as

$$u=Ku+f,(Ku)(x,y):=\int_a^b\int_c^d K(x,y,t,s,)\,u(t,s)\,dtds. \tag{24}$$

Theorem 2. Suppose that $K\in C^{r+1}(D\times D)$, $f\in C^{r+1}(D)$ and that the hypothesis of Lemma 1 are satisfied. Then, for any $(x,y)\in D$ we have:

$$\begin{gathered}(K_{hk}QV)(x,y)=\sum_{i=0}^{r}\sum_{j=0}^{r-i}\tfrac{h^ik^j}{i!j!}\Bigg\{R(B_i)\,S(B_j)\left[\tfrac{\partial^{i+j-2}}{\partial t^{i-1}\partial s^{j-1}}K(x,y,t,s,)\,V_{0,0}(t,s)\right]_{t=a,s=c}^{b\ d}\\ +\sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1}w_\mu\overline{w}_\upsilon\tfrac{B_\alpha(c_\mu)B_\beta(d_\upsilon)}{\alpha!\beta!}\cdot\\ \cdot\left[\tfrac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}K(x,y,t,s,)\,\overline{V}_{i-\alpha,j-\beta}(t,s,c_\mu,d_\upsilon)\right]_{t=a,s=c}^{b\ d}\Bigg\}+\\ +O\left(h^{r+1}+k^{r+1}\right).\end{gathered} \tag{25}$$

Proof. According to the definition (12) of K_{hk} we have:

$$\begin{gathered}(K_{hk}QV)(x,y)=hk\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1}w_\mu\overline{w}_\nu K(x,y,x_{m,\mu},y_{n,\upsilon})\cdot\\ \cdot QV(x_{m,\mu},y_{n,\upsilon})\end{gathered} \tag{26}$$

Using Lemma 1 we find

$$\begin{gathered}QV(x_{m,\mu},y_{n,\upsilon})=V_{0,0}(x_{m,\mu},y_{n,\upsilon})+\\ +\sum_{i=0}^{r}\sum_{j=0}^{r-i}h^ik^j\overline{V}_{i,j}(x_{m,\mu},y_{n,\upsilon},c_\mu,d_\upsilon)+O\left(h^{r+1}+k^{r+1}\right).\end{gathered}$$

Substituting this expression into (26) and using Lemmas 2 and 3 we have:

$$\begin{gathered}
(K_{hk}QV)(x,y) = hk\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1} w_\mu \overline{w}_\nu K(x,y,x_{m,\mu},y_{n,\upsilon})\cdot \\
\cdot V_{0,0}(x_{m,\mu},y_{n,\upsilon}) + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1} w_\mu \overline{w}_\nu hk \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} K(x,y,x_{m,\mu},y_{n,\upsilon})\cdot \\
\cdot \overline{V}_{i,j}(x_{m,\mu},y_{n,\upsilon},c_\mu,d_\upsilon) + O(h^{r+1}+k^{r+1}) + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \cdot \\
\cdot \left\{ \tfrac{R(B_i)S(B_j)}{i!j!}\left[\tfrac{\partial^{i+j-2}}{\partial t^{i-1}\partial s^{j-1}} K(x,y,t,s,)\,V_{0,0}(t,s)\right]_{t=a,s=c}^{b\ \ d} + \right. \\
+ \sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1} w_\mu \overline{w}_\nu \sum_{\alpha=0}^{r-i-j}\sum_{\beta=0}^{r-i-j-\alpha} h^\alpha k^\beta \tfrac{B_\alpha(c_\mu)B_\beta(d_\upsilon)}{\alpha!\beta!}\cdot \\
\left. \cdot \left[\tfrac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}} K(x,y,t,s,)\,\overline{V}_{i,j}(t,s,c_\mu,d_\upsilon)\right]_{t=a,s=c}^{b\ \ d} \right\} + O(h^{r+1}+k^{r+1})
\end{gathered}$$

Writing the above expression as polynomials in h and k we obtain the Theorem 2.

Now we choose $V_{0,0}(x,y) = u^*(x,y)$ (the exact solution of equation (23)) and $V_{i,j}\,(i \neq 0, j \neq 0)$ to satisfy the following linear Fredholm integral equations:

$$\begin{gathered}
V_{i,j}(x,y) - \int_a^b\int_c^d K(x,y,t,s)\,V_{i,j}(t,s)\,dtds = \\
= \tfrac{R(B_i)S(B_j)}{i!j!}\left[\tfrac{\partial^{i+j-2}}{\partial t^{i-1}\partial s^{j-1}} K(x,y,t,s)\,V_{0,0}(t,s)\right]_{t=a,s=c}^{b\ \ d} + \\
+ \sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1} w_\mu \overline{w}_\upsilon \tfrac{B_\alpha(c_\mu)B_\beta(d_\upsilon)}{\alpha!\beta!}\cdot \\
\cdot \left[\tfrac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\left(K(x,y,t,s)\,\overline{V}_{i-\alpha,j-\beta}(t,s,c_\mu,d_\upsilon) - \right.\right. \\
\left.\left. -\ (1 - sgn(\alpha+\beta))\,V_{i,j}(t,s)\right)\right]_{t=a,s=c}^{b\ \ d}.
\end{gathered} \tag{27}$$

From Theorem 2 we have:

$$V(x,y) - (K_{hk}QV)(x,y) = f(x,y) + O\left(h^{r+1}+k^{r+1}\right). \tag{28}$$

Theorem 3. Let $K \in C^{r+1}(D\times D)$, $f \in C^{r+1}(D)$ and u^* be the exact solution of (23). Then, for sufficiently large M and N, the difference between the iterated discrete spline Galerkin solution $\overline{z}^{hk}$ and u^* can be written as:

$$\begin{gathered}
\overline{z}^{hk}(x,y) - u^*(x,y) = \sum_{i=p}^{\left[\frac{r}{2}\right]} h^{2i} V_{2i,0}(x,y) + \sum_{j=q}^{\left[\frac{r}{2}\right]} k^{2j} V_{0,2j}(x,y) + \\
+ \sum_{i=p}^{\left[\frac{r}{2}\right]-q}\ \sum_{j=p}^{\left[\frac{r}{2}\right]-i} h^{2i}k^{2j} V_{2i,2j}(x,y) + O\left(h^{r+1}+k^{r+1}\right), (x,y) \in D
\end{gathered} \tag{29}$$

where $V_{i,j}\,(i \neq 0, j \neq 0)$ satisfy the equation (27).

Proof. Let denote by $\eta(x,y) := V(x,y) - \overline{z}^{hk}(x,y)$ for any $(x,y) \in D$. Substracting (18) from (28) we get:

$$(I - K_{hk}Q)\,\eta(x,y) = O\left(h^{r+1} + k^{r+1}\right) \tag{30}$$

The operator series $K_{hk}Q$ converge uniformly to K as $h \to 0$ and $k \to 0$. Because $(I-K)^{-1}$ exists and is uniformly bounded, it follows that $(I - K_{hk}Q)^{-1}$ exist and are uniformly bounded for all sufficiently small value h and k. So we have:

$$\overline{z}^{hk}(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j V_{i,j}(x,y) + O\left(h^{r+1} + k^{r+1}\right), (x,y) \in D$$

Thus, to complete the proof, it is easily to verify that $V_{i,j}(x,y) = 0$ if i is odd or $i \leq 2p-1$, j is odd or $j \leq 2q-1$. Now, coming back to the two-dimensional nonlinear Fredholm integral equation (1), we choose $W_{0,0}(x,y) = u^*(x,y)$ (the exact solution of (1)) and $W_{i,j}$ $(i \neq 0,\ or\ j \neq 0)$ the functions to satisfy the following linear Fredholm integral equations:

$$\begin{gathered}
W_{i,j}(x,y) - \int_a^b\int_c^d K_u(x,y,t,s,u^*(t,s))\,dtds = \\
= \tfrac{R(B_i)S(B_j)}{i!j!}\left[\tfrac{\partial^{i+j-2}}{\partial t^{i-1}\partial s^{j-1}} K(x,y,t,s,V_{0,0}(t,s))\right]_{t=a,s=c}^{b\ \ d} + \\
+ \sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\sum_{\mu=0}^{p-1}\sum_{\upsilon=0}^{q-1} w_\mu \overline{w}_\upsilon \tfrac{B_\alpha(c_\mu)B_\beta(d_\upsilon)}{\alpha!\beta!}\cdot \\
\cdot\left[\tfrac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\left(K_u(x,y,t,s,u^*(t,s))\cdot \overline{W}_{i-\alpha,j-\beta}(t,s,c_\mu,d_\upsilon) - \right.\right. \\
\left.\left. - (1 - sgn(\alpha+\beta))\,W_{i,j}(t,s) - F_{i-\alpha,j-\beta}(x,y,t,s,\xi,\eta)\right)\right]_{t=a,s=c}^{b\ \ d}
\end{gathered} \tag{31}$$

where $\overline{W}_{0,0}(x,y,t,s) = 0$ for $i \neq 0$, or $j \neq 0$,

$\overline{W}_{i,j}(x,y,t,s) := \sum\limits_{\mu=0}^{i}\sum\limits_{\upsilon=0}^{j} W^{(\mu,\upsilon)}(x,y)\cdot\Phi_\mu(t)\,\Psi_\upsilon(s)$ and

$$\begin{gathered}
F_{i,j}(x,y,t,s,\xi,\eta) := \sum_{p=2}^{i+j}\tfrac{1}{p!}\left(\tfrac{\partial}{\partial u}\right)^p K(x,y,t,s,u^*(t,s))\cdot \\
\cdot\left(\sum_{\alpha_1+\ldots\alpha_p=i}\ \sum_{\beta_1\ldots\beta_p=j}\ \prod_{n=1}^{p} W_{\alpha_n,\beta_n}(t,s,\xi,\eta)\right)
\end{gathered}$$

For the two-dimensional nonlinear Fredholm integral equation (1), similarly to Theorem 3 we obtain the following essential results.

Theorem 4. Let suppose that $r \geq \max(p,q)$ is an integer number, $K \in C^{r+1}(D\times D)$, $f \in C^{r+1}(D)$ and u^* is the exact solution of (1). If $\overline{z}^{hk}$ is the iterated spline discrete Galerkin solution, then for sufficiently large M and N we have the following error expression:

$$\overline{z}^{hk}(x,y) - u^*(x,y) = \sum_{i=p}^{\left[\frac{r}{2}\right]} h^{2i} W_{2i,0}(x,y) + \sum_{j=q}^{\left[\frac{r}{2}\right]} k^{2j} W_{0,2j}(x,y) +$$
$$+ \sum_{i=p}^{\left[\frac{r}{2}\right]-q} \sum_{j=q}^{\left[\frac{r}{2}\right]-i} h^{2i} k^{2j} W_{2i,2j}(x,y) + O\left(h^{r+1} + k^{r+1}\right), (x,y) \in D$$

where $W_{i,j}$ $(i \neq 0$ *or* $j \neq 0)$ satisfy the equation (31).

From the expression of the error given by the above Theorem, it follows directly that the iterated spline discrete Galerkin method possesses very good convergence properties.

4 Numerical example

Consider the following nonlinear Fredholm integral equation [9]

$$u(x,y) = \int_0^1 \int_0^1 \frac{x}{1+y}(1+t+s)\, u^2(t,s)\, dt ds + \frac{1}{(1+x+y)^2} - \frac{x}{6(1+y)},$$
$$(x,y) \in [0,1] \times [0,1]$$

whose exact solution is: $u^*(x,y) = \frac{1}{(1+x+y)^2}$.

The exact solution u^* will be approximated by iterated spline discrete Galerkin method in the spline space $S^{(-1)}_{p-1,q-1}$ with $p = q = 1$, i.e. the spline approximating space $S^{(-1)}_{0,0}$ is the piecewise constant finite element space.

We choose uniform partition with $M = N = 1, 2, 4, 8, 16, 32$ and with $h = k = \frac{1}{N}$, $x_{m,1} = x_m + c_1 h$, $y_{n,1} = y_n + d_1 h$; $(0 \leq m, n \leq N-1)$ with $c_1 = d_1 = \frac{1}{2}$, $w_1 = \overline{w}_1 = 1$.

The resulting nonlinear algebraic systems have been solved by a Newton method. Denoting by $\overline{z}^{hk}_m$ the approximating spline solution, by u^* the exact solution, by $e^{(m)}_N := \max\left\{u^*(x,y) - \overline{z}^{hk}(x,y)\, |: (x,y) \in D\right\}$ the errors, and by $\alpha^{(i)} := \log_2 \frac{e^{(i)}_N}{e^{(i)}_{2N}}$ an estimate of a convergence order we have obtained using the Theorem 4 the results contained in the following table:

N	$\mathbf{e_N^{(0)}}$	$\alpha^{(\mathbf{0})}$	$\mathbf{e_N^{(1)}}$	$\alpha^{(\mathbf{1})}$	$\mathbf{e_N^{(2)}}$	$\alpha^{(\mathbf{2})}$
1	6.124×10^{-2}	1.540	7.686×10^{-3}	3.133	4.250×10^{-4}	4.810
2	2.103×10^{-2}	1.831	8.779×10^{-4}	3.671	1.512×10^{-5}	5.685
4	5.93×10^{-3}	1.951	6.910×10^{-5}	3.912	2.945×10^{-7}	5.868
8	1.535×10^{-3}	1.988	4.595×10^{-6}	3.981	5.06×10^{-9}	
16	3.87×10^{-4}	1.998	2.941×10^{-7}			
32	9.71×10^{-5}					

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