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Some Inequalities for Certain Means in Two Arguments

Dumitru Acu

Abstract

In this paper the inequalities (1) are established.

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1

The means in two arguments are important for a variety of reasons (see [3]). They have been intensively studied. Power means are among the most extensively investigated means. The theory of Gini means is closely connected to the theory of generalized entropies, data compression and Information Theory (see [1], [4], [5]).

Let $0 < a < b$. The power mean is

$$M_p = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{\frac{1}{p}} & , \ p \neq 0 , \\ (ab)^{\frac{1}{2}} & , \ p = 0 ; \end{cases}$$

and the Gini mean, is

$$S_p = \begin{cases} \left(\dfrac{a^{p-1} + b^{p-1}}{a+b}\right)^{\frac{1}{p-2}} & , \ p \neq 2 , \\ (a^a b^b)^{\frac{1}{a+b}} & , \ p = 2 . \end{cases}$$

In the paper [4] the authors propose the following conjecture: the inequalities

$$(1) \qquad \frac{S_p}{M_p}\begin{cases} <1 & , \quad p\in(0,1) \\ =1 & , \quad p\in\{0,1\} \\ >1 & , \quad p\in(-\infty,0)\cup(1,\infty) \end{cases}$$

are satisfied.

In this paper we study this conjecture.

2

We prove the following results.

Theorem. *The following inequalities are satisfied*

$$\frac{S_p}{M_p}\begin{cases} <1 & , \quad p\in(0,1\cup(1,2)) \\ =1 & , \quad p\in\{0,1\} \\ >1 & , \quad p\in(-\infty,0)\cup[2,\infty)\ . \end{cases}$$

Proof. It is easy to verify this theorem for $p\in\{0,1\}$. The inequality $S_2\,/\,M_2>1$ has been proved in [6] and [4].

For $p\in\mathbb{R}-\{0,1,2\}$ we denote $t:=b\,/\,a>1$ and we have

$$\frac{S_p}{M_p}=\frac{\left(\dfrac{1+t^{p-1}}{1+t}\right)^{\frac{1}{p-2}}}{\left(\dfrac{1+t^p}{2}\right)^{\frac{1}{p}}}\ .$$

We consider the function

$$f(t)=\frac{p}{p-2}\ln\frac{1+t^{p-1}}{1+t}-\ln\frac{1+t^p}{2} \quad , \quad t>1\ .$$

Simple calculation give

$$f'(t)=\frac{p}{p-2}\cdot\frac{t^{2p-2}-(p-1)t^p+(p-1)t^{p-2}-1}{(1+t)(1+t^{p-1})(1+t^p)}$$

Denoting

$$g(t) = t^{2p-2} - (p-1)t^p + (p-1)t^{p-2} - 1 \quad , \quad t > 1 \ ,$$

we have

$$g'(t) = (p-1)t^{p-3}(2t^p - pt^2 + p - 2) \ .$$

If

$$h(t) = 2t^p - pt^2 + p - 2 \quad , \quad t > 1 \ ,$$

then we obtain

(2) $$h'(t) = 2p(t^{p-1} - 1) \ .$$

We consider four cases.

2.1. If $p > 2$ we have $h'(t) > 0$ for $t > 1$. This relations implies that the function h is increasing. From $h(1) = 0$ we obtain $h(t) > 0$ for $t > 1$. Then $g'(t) > 0$ for $t > 1$, where it results that the functions g is increasing. From $g(1) = 0$ we obtain $g(t) > 0$ for $t > 1$. Hence, $f'(t) > 0$ for $t > 1$ which implies that $f(t) > f(1) = 0$. From $f(t) > 0$ for $t > 1$, we obtain $S_p \,/\, M_p > 1$.

2.2. If $p \in (1, 2)$ then $h'(t) > 0$ for $t > 1$. We infer that h is an increasing function on $(1, \infty)$. From $h(1) = 0$ we find $h(t) > 0$ for $t > 1$. Then $g'(t) > 0$ for $t > 1$, where we find $g(t) > g(1) = 0$. Now, we obtain $f'(t) < 0$ for $t > 1$ which implies that $f(t) < f(1) = 0$. From $f(t) < 0$ for $t > 1$, we find $S_p \,/\, M_p < 1$.

2.3. If $p \in (0, 1)$, then we have $h'(t) < 0$, which implies that $h(t) < h(1) = 0$. From this inequality we conclude that $g'(t) > 0$ for $t > 1$. As above we obtain that $g(t) > g(1) = 0$. It is easy to observe that $f'(t) < 0$ which gives $f(t) < f(1) = 0$ for $t > 1$. Thus $S_p \,/\, M_p < 1$ is verified.

2.4. Finally, we consider $p < 0$. As above from (2) we obtain that $h'(t) > 0$ for $t > 1$. This implies that $h(t) > h(1) = 0$ which gives $g'(t) < 0$ for $t > 1$. From this inequality we have $g(t) < g(1) = 0$. Of course, it follows that the function f is increasing, where we have $f(t) > f(1) = 0$ for $t > 1$. From this inequality we conclude that $S_p \,/\, M_p < 1$ holds.

This completes the proof of the theorem.

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Department of Mathematics
Faculty of Sciences
University "Lucian Blaga" of Sibiu
Str.I.Raţiu, No.7,
2400 Sibiu, Romania
E-mail: *depmath@ulbsibiu.ro*