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Note on a class of delta operators

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Abstract

Let Q be a delta operator with basic set (p_n), and

$$A_n(x) = x(x+nb)^{n-1},\ n = 1, 2, ...$$

the Abel polynomials.

For each natural number $n, n \geq 1$ let us consider the delta operator defined on the algebra of polynomials, by

$$\alpha_{n,Q} = Q(Q+nbI)^{n-1},\ b \neq 0. \tag{1}$$

The purpose of this paper is to give a representation theorem for basic set (q_m) relative to $\alpha_{n,D}$, and we define a linear operator $t_{n,D}$ whence we obtain the $R_{n,a}$ and $\alpha_{n,D}$ operators (see [1], [2]).

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1

Theorem 1 *If $m = min(k, n)$, we have*

$$(\alpha_{n,Q}p_k)(x) = \frac{1}{n}\sum_{j=1}^{m}\frac{(-n)_j(-k)_j}{(j-1)!}(nb)^{n-1}p_{k-j}(x) \tag{2}$$

where $(a)_s = a(a+1)\cdot ... \cdot(a+s-1)$.

Proof.

$$A_n(x) = \sum_{j=0}^{n-1} \left(\begin{array}{c} n-1 \\ j \end{array} \right) (nb)^{n-j-1} x^{j+1} =$$

$$= \sum_{j=1}^{n} \left(\begin{array}{c} n-1 \\ j-1 \end{array} \right) (nb)^{n-j} x^{j}.$$

Hence

$$\alpha_{n,Q} = \sum_{j=1}^{n} \left(\begin{array}{c} n-1 \\ j-1 \end{array} \right) (nb)^{n-j} Q^{j},$$

whence

$$(\alpha_{n,Q} p_k)(x) = \sum_{j=1}^{m} \left(\begin{array}{c} n-1 \\ j-1 \end{array} \right) \frac{k!}{(k-j)!} (nb)^{n-j} p_{k-j}(x)$$

with $m = min(k,n)$.

Theorem 2 *If (q_m) is a basic sequence for the delta operator $\alpha_{n,D}$, then for $n > 0$*

$$q_m = \frac{1}{(nb)^{mn-m}\Gamma(mn-m)} \int_0^\infty e^{-t} t^{mn-m-1} x \left(x - \frac{1}{nb} t \right)^{m-1} dt. \tag{3}$$

Proof. We have

$$\alpha_{n,D} = D(D + nbI)^{n-1}$$

and

$$q_m = x(D + nbI)^{-mn+m} e_{m-1},$$

$$q_m(x) = x \frac{I}{(D + nbI)^{mn-m}} x^{m-1}.$$

Hence

$$q_m(x) = \frac{1}{(nb)^{mn-m}} \cdot \frac{1}{\Gamma(mn-m)} \int_0^\infty e^{-t} t^{mn-m-1} x \left(x - \frac{1}{nb} t \right)^{m-1} dt.$$

2

Ler R be a delta operator with two Sheffer sequences (r_n) and $(\widetilde{r}_n)$.

We note

$$T = e^{ax} D^m e^{-ax} = (D - aI)^m$$

and

$$t_n(x) = S(TE^b)^n \widetilde{r}_{n-1}(x)$$

where S is the linear operator defined by

$$S\widetilde{r}_n = r_{n+1},\ n = 0, 1, 2, ...$$

Theorem 3 *If m is natural number, $a, b \in \mathbb{R},\ a \neq 0$ and $R'_S, P'_S \in \prod_t^*$ where*

$$P^{-1} = TE^b$$

then

$$t_n(x) = t_n(a, b, m; \widetilde{r}_n; x) = Se^{ax} D^{nm} e^{-ax} \widetilde{r}_{n-1}(x + nb)$$

is a set of Sheffer polynomials.

Proof. It is used the theorem of [3].

For $m = 0$, we find

$$t_n(a, b, 0; \widetilde{r}_n; x) = S\widetilde{r}_{n-1}(x + nb)$$

and for $a = m = 1,\ b = 0$,

$$t_n(1, 0, 1; \widetilde{r}_n; x) = S(D - I)^n \widetilde{r}_{n-1}(x).$$

Now for $\widetilde{r}_n = r_n = x^n$ we have $S = X,\ (XP)(x) = xp(x)$, and

$$t_n(a, b, 0; x^n; x) = x(x + nb)^{n-1},$$

$$t_n(1, 0, 1; x^n, x) = x(D - I)^n x^{n-1}.$$

Hence from

$$t_n(x) = t_n(a, b, m; \widetilde{r}_n; x) = S(D - aI)^{mn} \widetilde{r}_{n-1}(x + nb)$$

we obtain the linear operator

$$t_{n,D} = t_n(D)$$

whence we find $R_{n,\alpha}$ and $\alpha_{n,D}$ operators, (see [1], [2]).

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