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On the exponential diophantine equations of the form $a^x - b^y c^z = \underline{+}1$

Dumitru Acu

Abstract

The aim of this paper is to obtain by elementary methods the solutions for some exponential diophantine equations of the form (1).

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1 Introduction

There have been many papers concerned with the solvability of the exponential diophantine equations

$$a^x - b^y c^z = \underline{+}1, \tag{1}$$

where the bases are given by positive integers and the exponents are unknown nonnegative integers. They have the applications to the theory of finite nonabelian simple groups (see [2], [3]).

Using only the congruences with a single modulus, in [2], [3], [1] the authors obtain all the solutions for some diophantine equations of the form (1):

$$\begin{array}{cc} 2^x - 3^y 7^z = \underline{+}1, & 3^x - 2^y 7^z = \underline{+}1; \\ 7^x - 2^y 3^z = \underline{+}1, & \text{(in [2])}; \\ 2^x - 3^y 5^z = \underline{+}1, & 2^x - 5^y 7^z = \underline{+}1; \\ 3^x - 2^y 5^z = \underline{+}1, & 5^x - 2^y 3^z = \underline{+}1; \\ 5^x - 2^y 7^z = \underline{+}1, & 7^x - 2^y 5^z = \underline{+}1 \text{ (in [3])}; \\ 2^x - 3^y 11^z = \underline{+}1, & 2^x - 3^y 13^z = \underline{+}1; \\ 2^x - 3^y 17^z = \underline{+}1, & 2^x - 3^y 19^z = \underline{+}1 \text{ (in [1])}. \end{array}$$

Chen L.S. and Wang Y.B. ([5]) find the solutions for the equations:

$$3^x - 2^y 5^z = -1 \text{ and } 5^x - 2^y 3^z = \underline{+}1.$$

in [5] using some elementary methods, Teng Y.A. proves that the equation

$$a^x - p_1^{y_1} p_2^{y_2} ... p_k^{y_k} = \underline{+}1.$$

where a is a positive integer with $a > 1$ and $p_1, p_2, ..., p_k$ are distinct primes with gcd $(a, p_1 p_2 ... p_k) = 1$, has only finitely many positive integer solutions $(x, y_1, ..., y_k)$.

If the number a, b, c are from the same parity, then for the equation (1) there is no a solution.

2 New results

Using the same method of the congruences with a single modulus (see [2] - [4]) we solve new diophantine equations of the type (1).

Theorem 1 . *The only solutions of the equation*

$$3^x - 2^y 11^z = 1$$

in nonnegative integers are: $(x, y, z) = (1, 1, 0), (2, 3, 0)$ *and* $(5, 1, 2)$.

Proof. For $z = 3$, using *mod 1331*, we find $x \equiv 0(mod\ 55)$. Now, if we use *mod 88573* we obtain a contradiction.

Suppose $y \geq 4$. Using *mod 16* we find $x \equiv 0$ *(mod 4)*. This yields a contradiction *mod 5*.

If z $= 0$ and $y \leq 3$, we find the solutions (x,y,z)= (1,1,0) and (2,3,0).

For z $= 1$ and $y \leq 3$ there is no solution.
If z $= 1$ and $y \leq 3$, we find the solution (x,y,z)= (5,1,2).

Theorem 2 . *The equation*

$$3^x - 2^y 11^z = -1$$

has in nonnegative integers only the solution $(x, y, z) = (0, 1, 0)$ *and* $(1, 2, 0)$.

Proof. Assume $z \geq 1$. Using *mod* 11, we obtain a contradiction.

If $y \geq 4$, using *mod* 16 we find a contradiction.

For $z = 0$ and $y \leq 3$ we obtain the solution $(x, y, z) = (0, 1, 0)$ and $(1, 2, 0)$.

Theorem 3 . *The solution to*

$$3^z - 2^y 13^z = 1$$

in the set of nonnegative integers are: $(x, y, x) = (1, 1, 0), (2, 3, 0)$ *and* $(3, 1, 1)$.

Proof. Suppose $z \geq 2$. Then using *mod* 169 we find $x \equiv 0(mod\ 39)$. Now we have a contradiction *mod* 5.

If $y \geq 4$, then using *mod* 16 we conclude that $x \equiv 0(mod\ 4)$. Hence, using *mod* 5, we have a contradiction.

If $z = 0$ and $y \leq 3$, we obtain the solutions $(x, y, z) = (1, 1, 0)$ and $(2, 3, 0)$.

For $z = 1$ and $y \leq 3$, we find the solutions $(x, y, x) = (3, 1, 1)$.

Theorem 4 . *The solutions of the equation*

$$3^x - 2^y 13^z = -1$$

in nonnegative integers are: $(x, y, x) = (0, 1, 0)$ *and* $(1, 2, 0)$.

Proof. For $z \geq 1$ we have a contradiction *mod* 13.

If $y \geq 4$, using *mod* 16, we obtain a contradiction.

For $z = 0$, and $y \leq 3$ we find only solutions from the theorem.

Theorem 5 . *The only solution to each of the following equations*

(2) $$3^x - 2^y 17^z = 1$$

and

(3) $$3^x - 2^y 19^z = 1$$

in nonnegative integers are: $(x, y, z) = (1, 1, 0)$ *and* $(2, 3, 0)$.

Proof. (2) Suppose $z \geq 1$. Then using *mod* 17, we find $x \equiv 0(mod\ 16)$. Now, we have a contradiction *mod*5.

If $y \geq 4$, then using *mod* 16 we conclude $x \equiv 0(mod\ 18)$. Hence, using *mod* 5 we obtain a contradiction.

For $z = 0$ and $y \leq 3$, we find only solutions from the theorem.

(3) Assume $z \geq 1$. Using *mod* 19, we find $x \equiv 0(mod\ 18)$. Thus, examining *mod* 7 we obtain a contradiction.

If $y \geq 4$, then using *mod* 7 we have a contradiction.

For $z = 0$ and $y \leq 3$, we find only solutions from the theorem.

Theorem 6 . *Each of the following diophantine equations*

(4) $$3^x - 2^y 17^z = -1$$

and

(5) $$3^z - 2^y 19^z = -1$$

has in nonnegative integers only the solutions $(x, y, x) = (0, 1, 0)$ *and* $(1, 2, 0)$.

Proof. (4) Assume $z \geq 1$. Using mod 17, we obtain $x \equiv 8(mod\ 16)$. Thus, using $mod 193$ we have a contradiction.

If $y \geq 4$, using mod 16 we obtain a contradiction.

For $z = 0$ and $y \leq 3$, we find the solutions $(x, y, x) = (0, 1, 0)$ and $(1, 2, 0)$.

(5) Suppose $z \geq 1$. Using mod 19, we find $x \equiv 9(mod\ 18)$. Hence, examining mod 7 we have a contradiction.

For $y \geq 4$, using mod 16 we obtain a contradiction.

If $z = 0$ and $y \leq 3$, we have only solutions from the theorem.

Theorem 7 . *The only solution to each of the following equations*

(6) $$5^x - 2^y 11^z = 1$$

(7) $$5^x - 2^y 13^z = 1$$

(8) $$5^x - 2^y 17^z = 1$$

and

(9) $$5^x - 2^y 19^z = 1$$

in nonnegative integers is: $(x, y, x) = (1, 2, 0)$.

Proof.

(6) For $z \geq 1$, using mod 11, it results $x \equiv 0(mod\ 5)$. Thus, mod 71 yields a contradiction.

If $y \geq 3$, then using mod 8 we conclude that $x \equiv 0(mod\ 2)$. Now, with mod 3 we obtain a contradiction.

(7) If $z \geq 1$, using mod 13, we conclude that $x \equiv 0(mod\ 4)$. Then, using mod 3 we obtain a contradiction.

For $y \geq 3$, examining mod 8 we conclude that x is even. This yields a contradiction mod 3.

(8) If $z \geq 1$, using mod 17, we find $x \equiv 0(mod\ 16)$. Then, examining mod 3 we obtain a contradiction.

For $y \geq 3$, using mod 8 we conclude that x is even. This yields a contradiction mod 3.

(9) Assume $z \geq 1$. With *mod* 19, we find $x \equiv 0(mod\ 9)$. Thus, we have a contradiction *mod* 31.

If $y \geq 3$, using *mod* 8 we obtain $x \equiv 0(mod\ 2)$. This yields a contradiction *mod* 3.

Theorem 8 . *Each of the diophantine equations*

$$5^x - 2^y 11^z = -1 \tag{10}$$

$$5^x - 2^y 17^z = -1 \tag{11}$$

and

$$5^x - 2^y 19^z = -1 \tag{12}$$

has only the solution $(x, y, x) = (0, 1, 0)$.

Proof.

(10) We use *mod* 11 for $z \geq 1$ and *mod* 4 for $y \geq 2$. In each case we have a contradiction.

If $z \geq 1$, then using *mod* 8 we conclude that $x \equiv 0(mod\ 2)$. Now, with *mod* 3 we obtain a contradiction.

(11) If $z \geq 1$, using *mod* 17, we conclude that $z \equiv 8(mod\ 16)$. Thus, we have a contradiction *mod* 11489.

For $y \geq 2$ we obtain a contradiction *mod* 4.

(12) If $z \geq 1$, we examine *mod* 19 and we find a contradiction.

For $y \geq 2$, with *mod* 4 we obtain a contradiction.

Theorem 9 . *The only solutions to*

$$5^x - 2^y 13^z = -1$$

in nonnegative integers are $(x, y, x) = (0, 1, 0)$ *and (2,1,1).*

Proof.

Suppose $z \geq 2$. Thus, using *mod* 169 we conclude that $x \equiv 26(mod\ 52)$. Hence, examining *mod* 4408627572144277 we obtain a contradiction.

If $y \geq 2$, we have a contradiction *mod* 4.

Theorem 10 . *Let p_1, p_2 be distinct primes and different from 3. Then the equation*

$$7^x - p_1^y p_2^z = 1$$

has no solution in nonnegative integers.

Proof.

Using *mod* 3 we obatin a contradiction.

Corollar 1 . *The equations*

$$7^x - 2^y 11^z = 1, \qquad 7^x - 2^y 13^z = 1$$

$$7^x - 2^y 17^z = 1 \text{ and } 7^x - 2^y 19^z = 1$$

have no solutions in nonnegative integers.

Theorem 11 . *Each of the diophantine equations*

$$7^x - 2^y 11^z = -1 \tag{13}$$

$$7^x - 2^y 13^z = -1 \tag{14}$$

$$7^x - 2^y 17^z = -1 \tag{15}$$

and

$$7^x - 2^y 19^z = -1 \tag{16}$$

has only the solutions $(x, y, x) = (0, 1, 0)$ and (1,3,0).

Proof.

(13) Assume $z \geq 1$. Then using *mod* 11 we conclude that $x \equiv 5(mod\ 10)$. Thus, examining *mod* 191 we obtain a contradiction. If $y \geq 5$, then *mod* 32 yields a contradiction.

(14) Suppose $z \geq 1$. Then with *mod* 13, we find $x \equiv 6(mod\ 12)$.

For $y \geq 5$ we obtain a contradiction *mod* 32.

(15) If $x \geq 1$, using *mod* 17 we conclude that $x \equiv 8(mod\ 16)$. Thus, examining *mod* 169553 we obtain a contradiction. The case $y \geq 5$ yields a contradiction *mod* 32.

(16) If $z \geq 2$, using *mod* 361, we have a contradiction. For $y \geq 5$ we obtain a contradiction *mod* 32.

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Department of Mathematics
University "Lucian Blaga" of Sibiu,
Str. Dr. I. Ratiu Nr.7,
2400 Sibiu, Romania
E-mail address: *depmath@ulbsibiu.ro*