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On a second-order nonlinear differential subordination I

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Abstract

We find conditions on the complex-valued functions A, B, C, D and E in the unit disc U such that the differential inequality

$$\text{Re } [A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)] > 0$$

implies $\text{Re } p(z) > 0$, where $p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \dots$ is analytic in U.

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1 Introduction and preliminaries

In [1] chapter IV the authors have analyzed a second-order linear differential subordination

$$(1) \qquad A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where A, B, C, D and h are complex-valued functions. A more general version of (1) is given by:

$$(2) \qquad A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega,$$

where $\Omega \subseteq \mathbb{C}$.

In this paper we shall extend this problem by considering a second-order nonlinear differential subordination given by

$$(3) \quad A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \prec h(z).$$

A more general version of (3) is given by:

$$(4) \quad A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \in \Omega,$$

where $\Omega \subseteq \mathbb{C}$.

Conditions on the complex-valued functions A, B, C, D, E and h will be determined so that the differential subordinations given by (3) and (4) will have dominants and even best dominants.

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Definition. [1, Definition 2.3.a. p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r, s, t; z) \notin \Omega \tag{5}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\mathrm{Re}\ \frac{t}{s} + 1 \geq m\mathrm{Re}\ \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right],$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is conformal mapping of U onto Ω we denote this class by $\Psi_n[h, q]$.

If $\Omega = \Delta = \{w \in \mathbb{C} : \ \mathrm{Re}\ w > 0\}$, $q(z) = \dfrac{1+z}{1-z}$, $q \in Q$, satisfies $q(U) = \Delta$, $q(0) = 1$, $E(q) = \{1\}$, the class of admissible functions $\Psi_n[\Omega, q]$ is denoted by $\Psi_n[\Omega, 1] = \Psi_n\{1\}$, the condition of admissibility (5) becomes

$$\psi(\rho i, \sigma, \mu + \nu i; z) \notin \Omega, \tag{A}$$

when $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\dfrac{n}{2}(1+\rho^2)$, $\sigma + \mu \leq 0$, $z \in U$, and $n \geq 1$.

Lemma B. [1, Theorem 2.3.i p. 35] *Let $\psi \in \Psi_n\{1\}$. If $p \in \mathcal{H}[1, n]$ and*

$$\mathrm{Re}\ [\psi(p(z), zp'(z), z^2 p''(z); z)] > 0$$

then

$$\mathrm{Re}\ p(z) > 0.$$

More general forms of this lemma can be found in [1] p. 35.

In this paper we shall analyze the case when $\Omega = \Delta = \{w \in \mathbb{C} : \ \mathrm{Re}\ w > 0\}$, and $h(z) = q(z) = \dfrac{1+z}{1-z}$, $z \in U$.

2 Main results

Theorem. *Let n be a positive integer and $A(z) \equiv A \geq 0$. Suppose that the functions $B, C, D, E : U \to \mathbb{C}$ satisfy*

$$(6) \quad \begin{cases} \text{Re } B(z) \geq A \;;\; \text{Re } [nB(z) + 2C(z)] \geq nA \\ [\text{Im } D(z)]^2 \leq \text{Re } [nB(z) + 2C(z) - nA] \cdot \text{Re } [nB(z) - 2E(z) - nA] \end{cases}$$

If $p \in \mathcal{H}[1, n]$ and if

$$(7) \quad \text{Re } [Az^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)] > 0$$

then

$$\text{Re } p(z) > 0, \quad z \in U.$$

Proof. We let $r = p(z)$, $s = zp'(z)$, $z^2p''(z) = t$, $z \in U$, $r, s, t \in \mathbb{C}$. If we let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be given by

$$(8) \qquad \psi(r, s, t; z) = At + B(z)s + C(z)r^2 + D(z)r + E(z),$$

then the conclusion of the theorem will follow from Lemma B.

For $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1 + \rho^2)$, $\sigma + \mu \leq 0$ and $z \in U$, by using (6) we obtain

$$\text{Re } \psi(\rho i, \sigma, \mu + \nu i; z) = \text{Re } [A(\mu + \nu i) + \sigma B(z) + (\rho i)^2 C(z) + D(z)\rho i + E(z)] =$$

$$= A\mu + \sigma \text{Re } B(z) - \rho^2 \text{Re } C(z) - \rho \text{Im } D(z) + \text{Re } E(z) \leq$$

$$\leq -A\sigma + \sigma \text{Re } B(z) - \rho^2 \text{Re } C(z) - \rho \text{Im } D(z) + \text{Re } E(z) \leq$$

$$\leq -\frac{1}{2}[\text{Re } (nB(z) + 2C(z)) - nA]\rho^2 - \text{Im } D(z)\rho - \frac{1}{2}\text{Re } [nB(z) - nA - 2E(z)] \leq 0.$$

Hence, the function ψ given by (8) verifies the admissibility condition (A). Since $h(0) = \psi(1, 0, 0, 0)$ we have that $\psi \in \Psi_n\{1\}$. By using Lemma B we have that Re $p(z) > 0$. $\square$

For $C(z) = 0$ we obtain Theorem 4.1.a [1] p. 188.

If $A(z) = A > 0$, $E(z) = -C(z)$ then we obtain the following:

Corollary. *Let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy:*

$$\begin{cases} \text{Re } B(z) \geq A \\ |\text{Im } D(z)| \leq \text{Re } [nB(z) + 2C(z) - nA] \end{cases}$$

If $p \in \mathcal{H}[1, n]$ and if

$$\text{Re } \{Az^2p''(z) + B(z)zp'(z) + C(z)[p^2(z) - 1] + D(z)p(z)| > 0$$

then

$$\text{Re } p(z) > 0, \quad z \in U.$$

If $A = 0$ and $C(z) \equiv 0$, then Corollary reduces to a particular form of Corollary 4.1.a.1 [1, p. 189].

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

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