General Mathematics Vol. 9, No. 3–4 (2001)

On a first order nonlinear differential subordination

Georgia Irina Oros

Abstract

We find conditions on the complex-valued functions B, C, D in unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

2000 Mathematical Subject Classification: 30C80.
Keywords: differential subordination, dominant.

1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Lemma A. [1] (Lemma 2.2.d p. 24) *Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

(iii) $\mathrm{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq \mathrm{Re}\ \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where B, C, D and h are complex-valued functions in the unit disc U. A more general version of (1) is given by

$$B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{C}$.

In [2] we extended this problem by considering a first order nonlinear differential subordination

$$B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \prec h(z) \tag{3}$$

A more general version of (3) is given by:

$$B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z) \in \Omega \tag{4}$$

where $\Omega \subseteq \mathbb{C}$.

The general problem is to find conditions on the complex-valued functions B, C, D and h such that the differential subordination given by (3) and (4) will have dominants and even best dominants.

In this paper we consider the first order nonlinear differential subordination (4) in which $\Omega = \{w;\ |w| < M\}$ and $E(z) \equiv 0$. Given the functions B, C, D and the constant M, our problem is to find a constant N such that, for $p \in \mathcal{H}[0, n]$, the differential inequality

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies $|p(z)| < N$.

2 Main results

The results in [2] can certainly be used in the special case when $E(z) \equiv 0$. However, in this case we can improve those results by the following theorem:

Theorem. *Let $M > 0$ and let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy*

$$(5)\quad \begin{cases} (i) \quad n|B(z)| - |D(z)| \geq 2\sqrt{M|C(z)|}, \\ (ii) \quad N = \\ = \sup\limits_{|z|<1} \dfrac{2M}{n|B(z)| - |D(z)| + \sqrt{[n|B(z)| - |D(z)|]^2 - 4M|C(z)|}} < \infty, \end{cases}$$

If $p \in \mathcal{H}[0, n]$ and

$$(6)\qquad |B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

then

$$|p(z)| < N.$$

Proof. Let $p \in \mathcal{H}[0,n]$, $p(z) = a_n z^n + a_{n+1}z^{n+1} + \ldots$, $z \in U$.

$$W(z) = B(z)zp'(z) + C(z)p^2(z) + D(z)p(z), \tag{7}$$

From (7) and (6) we have

$$|W(z)| < M, \ z \in U. \tag{8}$$

Assume that $|p(z)| \not< N$, which is equivalent with $p(z) \not\prec Nz \equiv q(z)$.

According to Lemma A, with $q(z) = Nz$, there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$, $\theta_0 \in [0, 2\pi)$, $\zeta \in \partial U$, $|\zeta| = 1$ and $m \geq n$, such that $p(z_0) = N\zeta$ and $z_0 p'(z_0) = mN\zeta$. Using these conditions in (7) we obtain for $z = z_0$

$$|W(z_0)| = |B(z_0)mN\zeta + C(z_0)N^2\zeta^2 + D(z_0)N\zeta| =$$

$$= N[|B(z_0)m + C(z_0)N\zeta + D(z_0)|] \geq N[|B(z_0)m + C(z_0)N\zeta| - |D(z_0)|] \geq$$

$$\geq N[|B(z_0)|m - N|C(z_0)| - |D(z_0)|] \geq$$

$$\geq -N^2|C(z_0)| + N[n|B(z_0)| - |D(z_0)|].$$

We shall find N such that

$$|W(z_0)| - M \geq 0$$

The condition $|W(z_0)| - M \geq 0$ is equivalent to:

$$-N^2|C(z_0)| + N[n|B(z_0)| - |D(z_0)|] - M \geq 0. \tag{9}$$

Since $-|C(z_0)| < 0$ inequality (9) holds if and only if

$$n|B(z_0)| - |D(z_0)| \geq \sqrt{4M|C(z_0)|}$$

and by (i) and (ii) we deduce that the condition (9) holds for all $z_0 \in U$.

Since (9) contradicts (8) we obtain the desired result $|p(z)| < N$. $\Box$

If $n = 1$, $B(z) = 10 + 8i$, $C(z) = 2 + 2\sqrt{3}i$, $D(z) = 1 - i$, $M = 4$, then

$$N = \frac{8}{2\sqrt{41} - \sqrt{2} + \sqrt{102 - 4\sqrt{82}}}.$$

Theorem can be written as the following:

Corollary 1. *If* $p \in \mathcal{H}[0,1]$ *and*

$$|(10+8i)zp'(z)+(2+2\sqrt{3}i)p^2(z)+(1-i)p(z)|<4$$

then

$$|p(z)|<\frac{8}{2\sqrt{41}-\sqrt{2}+\sqrt{102-4\sqrt{82}}}.$$

If $n=2$, $B(z)=4+6i$, $C(z)=2i$, $D(z)=1+i$, $M=10$, *then*

$$N=\frac{20}{4\sqrt{13}-\sqrt{2}+\sqrt{90-8\sqrt{26}}}.$$

Theorem can be written as the following:

Corollary 2. *If* $p \in \mathcal{H}[0,1]$ *and*

$$|(4+6i)zp'(z)+2ip^2(z)+(1+i)p(z)|<10$$

then

$$|p(z)|<\frac{20}{4\sqrt{13}-\sqrt{2}+\sqrt{90-8\sqrt{26}}}.$$

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Gh. Oros and Georgia Irina Oros, *On a particular first-order nonlinear differential subordination I*, (preprint).

Faculty of Mathematics and Computer Sciences
Babeş-Bolyai University
3400 Cluj-Napoca, Romania