Georgian Mathematical Journal
1(1994), No. 1, 1-8

ON CONJUGACY OF HIGH-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is shown that the differential equation

$$u^{(n)} = p(t)u,$$

where $n \geq 2$ and $p : [a, b] \to \mathbb{R}$ is a summable function, is not conjugate in the segment $[a, b]$, if for some $l \in \{1, \dots, n-1\}$, $\alpha \in]a, b[$ and $\beta \in]\alpha, b[$ the inequalities

$$n \geq 2 + \frac{1}{2}(1 + (-1)^{n-l}), \quad (-1)^{n-l}p(t) \geq 0 \quad \text{for} \quad t \in [a, b],$$

$$\int_\alpha^\beta (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)!\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)},$$

hold.

Consider the differential equation

$$u^{(n)} = p(t)u, \tag{1}$$

where $n \geq 2$, $p \in L_{loc}(I)$, and $I \subset \mathbb{R}$ is an interval.

The following definitions will be used below.

Equation (1) is said to be conjugate in I if there exists a nontrivial solution of this equation with at least n zeroes (each zero counted accordingly to its multiplicity) in I.

Let $l \in \{1, \dots, n-1\}$. Equation (1) is said to be $(l, n-l)$ conjugate in I if there exists a nontrivial solution u of this equation satisfying

$$u^{(i)}(t_1) = 0 \quad (i = 0, \dots, l-1),$$
$$u^{(i)}(t_2) = 0 \quad (i = 0, \dots, n-l-1),$$

with $t_1,\ t_2 \in I$ and $t_1 < t_2$.

Suppose first that $-\infty < a < b < +\infty$ and $p \in L([a, b])$.

1991 *Mathematics Subject Classification.* 34C10.

Lemma. *Let $a < \alpha < \beta < b$. Then the Green's function G of the problem*

$$\begin{aligned} &u^{(n)}(t) = 0 \quad \text{for } t \in [a,b],\\ &u^{(j)}(a) = 0 \quad (j = 0, \dots, l-1),\\ &u^{(j)}(b) = 0 \quad (j = 0, \dots, n-l-1), \end{aligned}$$

satisfies the inequality

$$\begin{aligned} &(-1)^{n-l}G(t,s) >\\ &> \frac{(b-\beta)(\alpha-a)(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times\\ &\times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for } \alpha \le t < s \le \beta. \end{aligned} \tag{2}$$

Proof. The function G can be written in the form

$$G(t,s) = \begin{cases} \displaystyle\sum_{i=n-l+1}^{n} (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for} \quad a \le s < t \le b,\\ \displaystyle -\sum_{i=1}^{n-l} (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for} \quad a \le t \le s \le b, \end{cases}$$

where

$$x_i(t) = \frac{(t-a)^{n-i}(b-t)^{i-1}}{(i-1)!(b-a)^{n-i}}.$$

It is easy to verify that for any fixed $s \in]a,b[$ the function $\frac{(-1)^{n-l}G(\cdot,s)}{x_{n-l}(\cdot)x_{l+1}(s)}$ decreases on $]a,b[$ and the function $\frac{(-1)^{n-l}G(\cdot,s)}{x_{n-l+1}(\cdot)x_{l}(s)}$ increases on $]a,b[$. Thus

$$(-1)^{n-l}G(t,s) \ge (-1)^{n-l}G(s,s)\frac{x_{n-l}(t)}{x_{n-l}(s)} \quad \text{for} \quad t \le s. \tag{3}$$

Taking into account that

$$\begin{aligned} (-1)^{n-l}G(s,s) &= (-1)^{n-l-1}\sum_{i=1}^{n-l}(-1)^{i-1}x_i(s)x_{n-i+1}(s) =\\ &= \frac{(s-a)^{n-1}(b-s)^{n-1}}{(b-a)^{n-1}} \sum_{i=1}^{n} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \end{aligned}$$

and

$$\frac{x_{n-l}(t)}{x_{n-l}(s)} = \frac{(t-a)^{l}(b-t)^{n-l-1}}{(s-a)^{l}(b-s)^{n-l-1}},$$

from the inequality (3) we deduce

$$\begin{aligned}
&(-1)^{n-l}G(t,s) \geq \\
&\geq \frac{(s-a)^{n-l-1}(b-s)^{l}(t-a)^{l}(b-t)^{n-l-1}}{(b-a)^{n-1}} \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} > \\
&> (b-\beta)(\alpha-a)\frac{(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times \\
&\times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for} \quad \alpha \leq t < s \leq \beta. \qquad \square
\end{aligned}$$

Theorem 1. *Let $l \in \{1,\dots,n-1\}$,*

$$n \geq 2 + \frac{1+(-1)^{n-l}}{2} \quad \textit{and} \quad (-1)^{n-l}p(t) \geq 0 \quad \textit{for} \quad t \in [a,b]. \tag{4}$$

If, in addition, there exist $\alpha,\beta \in]a,b[$ such that $a < \alpha < \beta < b$ and

$$\int_{\alpha}^{\beta} (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)!\frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)}, \tag{5}$$

then Eq. (1) is $(l, n-l)$ conjugate in $[a,b]$.

Note that analogous results are given in [3,5] for the case where $n = 2$.

Proof. Put $p(t) = 0$ for $t > b$ and consider Eq. (1) in the interval $[a,+\infty[$. For any $\gamma > a$, let u_γ be the solution of (1) satisfying

$$\begin{aligned}
&u_\gamma^{(i)}(a) = 0 \qquad (i = 0,\dots,l-1), \\
&u_\gamma^{(i)}(\gamma) = 0 \qquad (i = 0,\dots,n-l-2), \\
&\sum_{i=0}^{n-1} |u_\gamma^{(i)}(a)| = 1, \quad \max\{u_\gamma(t) : a \leq t \leq \gamma\} > 0.
\end{aligned}$$

Suppose now that in spite of the statement of the theorem Eq. (1) is not $(l,n-l)$ conjugate in $[a,b]$.

Note that if $\gamma \in]a,b]$, then $u_\gamma(t) > 0$ for $t \in]a,\gamma[$ and $(-1)^{n-l-1} \times u_\gamma^{(n-l-1)}(\gamma) > 0$. Indeed, if it is not so, there exists $t_0 \in]a,\gamma[$ such that $u_\gamma(t_0) = 0$. Let $\gamma_0 = \inf\{\gamma > a : u_\gamma(t) = 0$ for a certain $t \in]a,\gamma[\}$. Then $u_{\gamma_0}(t) > 0$ for $t \in]a,\gamma_0[$ and

$$\begin{aligned}
&u_{\gamma_0}^{(i)}(a) = 0 \qquad (i = 0,\dots,l-1), \\
&u_{\gamma_0}^{(i)}(\gamma_0) = 0 \qquad (i = 0,\dots,n-l-1),
\end{aligned}$$

which contradicts our assumption.

Let $\gamma^0 = \sup\{\gamma > b : u_\gamma(t) > 0 \quad \text{for} \quad t \in]a, \gamma[\}$. Consider first the case where $\gamma^0 = +\infty$. There exists the sequence $\{\gamma_k\}_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty} \gamma_k = +\infty, \qquad \lim_{k\to+\infty} u_{\gamma_k}(t) = u_0(t)$$

where u_0 is the solution of Eq. (1). Show that

$$u_0(t) > 0 \quad \text{for} \quad t > a. \tag{6}$$

It is clear that $u_0(t) \geq 0$ for $t > a$. If now $u_0(t_*) = 0$ for some $t_* > a$, then for any k large enough the function u'_{γ_k} will have at least one zero in $]a, \gamma_k[$. Taking into account the multiplicities of zeroes of u_{γ_k} in a and γ_k, it is easy to show that $u_{\gamma_k}^{(n-1)}$ has at least two zeroes in $]a, \gamma_k[$. Hence $u_{\gamma_k}^{(n)}$ changes sign in this interval and this is impossible.

Thus inequality (6) is proved. This inequality and the results of [1] imply that there exist $l_0 \in \{1, \dots, n\}$ ($l - l_0$ is even) and $t_1 > b$ such that

$$\begin{aligned} u_0^{(i)}(t) > 0 \quad &\text{for} \quad t \geq t_1 \quad (i = 0, \dots, l_0 - 1), \\ (-1)^{i+l_0} u_0^{(i)}(t) \geq 0 \quad &\text{for} \quad t \geq t_1 \quad (i = l_0, \dots, n). \end{aligned} \tag{7}$$

Clearly,

$$\begin{aligned} (-1)^{i+l_0} u_0^{(i)}(t) \geq 0 \quad &\text{for} \quad t \geq a \quad (i = l_0, \dots, n), \\ (-1)^{i+l_0} u_0^{(i)}(a) > 0 \quad & \qquad\qquad\quad (i = l_0, \dots, n-1). \end{aligned} \tag{8}$$

Hence $l \in \{1, \dots, l_0\}$.

Suppose that $l < l_0$. Then for any k large enough we have $\gamma_k > t_1$, $u_{\gamma_k}^{(i)}(t_1) > 0 (i = 0, \dots, l_0 - 1)$. This means that the function $u_{\gamma_k}^{(i)}$ has at least one zero in $]t_1, \gamma_k[$. Taking into account the multiplicity of zero in γ_k, it is easy to see that $u_{\gamma_k}^{(n-1)}$ has at least two zeroes in $]t_1, \gamma_k[$, and $u_{\gamma_k}^{(n)}$ changes sign in this interval. But this is impossible. Thus $l = l_0$.

As $l = l_0$, inequalities (7) and (8) imply

$$\begin{aligned} (-1)^{i+l} u_0^{(i)}(t) \geq 0 \quad &\text{for} \quad t \geq a \quad (i = l, \dots, n), \\ u_0^{(i)} > 0 \quad &\text{for} \quad t > a \quad (i = 0, \dots, l - 1). \end{aligned}$$

Let

$$v(t) = u_0^{(l-1)}(t) - \sum_{j=l}^{n-1} \frac{(-1)^{j-l}}{(j-l+1)!} (t-a)^{j-l+1} u_0^{(j)}(t);$$

then

$$v'(t) = \frac{(-1)^{n-l}}{(n-l)!} (t-a)^{n-l} u_0^{(n)}(t) \geq 0 \quad \text{for} \quad t \geq a.$$

Hence

$$\begin{aligned} u_0^{(l-1)}(t) \geq \sum_{j=l}^{n-1} \frac{(-1)^{j-l}(t-a)^{j-l+1}u_0^{(j)}(t)}{(j-l+1)!} \geq \\ \geq \frac{(t-a)(-1)^{n-l}}{(n-l)!}\int_t^{+\infty}(s-a)^{n-l-1}p(s)u_0(s)ds \quad \text{for} \quad t \geq a. \end{aligned} \tag{9}$$

Denote $\rho_i(t) = iu_0^{(l-i)}(t) - (t-a)u_0^{(l-i+1)}(t)$ for $t \geq a$ $(i = 0, 1, \dots, l)$. Then $\rho_i'(t) = \rho_{i-1}(t)$ for $t \geq a$ $(i = 1, \dots, l)$. Since $\rho_0(t) = -(t-a)u_0^{(l+1)}(t) \geq 0$ for $t \geq a$ and $\rho_i(a) = 0$ $(i = 1, \dots, l)$, we have $\rho_i(t) \geq 0$ for $t \geq a$ $(i = 0, 1, \dots, l)$. This implies

$$u_0(t) \geq \frac{(t-a)^{l-1}}{l!}u_0^{(l-1)}(t). \tag{10}$$

From (9) and (10) we obtain

$$1 \geq \frac{(t-a)}{l!(n-l)!}\int_t^{\beta}(s-a)^{n-2}|p(s)|ds \quad \text{for} \quad t \geq a,$$

which contradicts (5). The case $\gamma^0 = +\infty$ is thus eliminated.

Now consider the case where $\gamma^0 < +\infty$. As we have already noted, $\gamma^0 > b$, $u_{\gamma^0}(t) > 0$ for $t \in]a, \gamma^0[$ and

$$\begin{aligned} u_{\gamma^0}^{(i)}(a) = 0 \qquad (i = 0, \dots, l-1), \\ u_{\gamma^0}^{(i)}(\gamma^0) = 0 \qquad (i = 0, \dots, n-l-1). \end{aligned} \tag{11}$$

Hence

$$u_{\gamma^0}(t) = \int_a^{\gamma^0} G(t,s)p(s)u_{\gamma^0}(s)ds,$$

where G is the Green's function of the boundary value problem (11) for the equation $u^{(n)} = 0$.

Let $t_0 \in]\alpha, \beta[$ be such that

$$\frac{u_{\gamma^0}(t)}{(t-a)^{l-1}(\gamma^0-t)^{n-l-1}} \geq \frac{u_{\gamma^0}(t_0)}{(t_0-a)^{l-1}(\gamma^0-t_0)^{n-l-1}} \quad \text{for} \quad t \in [\alpha, \beta]. \tag{12}$$

Then from the lemma and the inequality (12) it follows that

$$
\begin{aligned}
u_{\gamma^0}(t_0) \geq (\gamma^0-\beta)(\alpha-a)\sum_{i=1}^{n-l}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!}\times \\
\times\int_\alpha^\beta \frac{(s-a)^{n-2}(\gamma^0-s)^{n-2}}{(\gamma^0-a)^{n-1}}|p(s)|ds u_{\gamma^0}(t_0) > \\
> \frac{(b-\beta)(\alpha-a)}{(b-a)^{n-1}}\sum_{i=1}^{n-l}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!}\times \\
\times\int_\alpha^\beta (s-a)^{n-2}(b-s)^{n-2}|p(s)|ds u_{\gamma^0}(t_0).
\end{aligned}
\tag{13}
$$

Since $\sum\limits_{i=1}^{n-l}\frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \geq \frac{1}{l!(n-l)!}$ the inequality (13) contradicts (5). □

Denote

$$
\begin{aligned}
\mu_n^- &= \min\{l!(n-l)! : l \in \{1,\dots,n-1\},\ \ n-l \quad odd\},\\
\mu_n^+ &= \min\{l!(n-l)! : l \in \{1,\dots,n-1\},\ \ n-l \quad even\}.
\end{aligned}
$$

It is clear that

$$
\mu_n^- = \begin{cases} (\frac{n}{2}-1)!(\frac{n}{2}+1)! & \text{for} \quad n\equiv 0 \pmod 4,\\ \big[(\frac{n}{2})!\big]^2 & \text{for} \quad n\equiv 2 \pmod 4,\\ (\frac{n-1}{2})!(\frac{n+1}{2})! & \text{for} \quad n\equiv 1 \pmod 2, \end{cases}
$$

$$
\mu_n^+ = \begin{cases} \big[(\frac{n}{2})!\big]^2 & \text{for} \quad n\equiv 0 \pmod 4,\\ (\frac{n}{2}-1)!(\frac{n}{2}+1)! & \text{for} \quad n\equiv 2 \pmod 4,\\ (\frac{n-1}{2})!(\frac{n+1}{2})! & \text{for} \quad n\equiv 1 \pmod 2. \end{cases}
$$

Corollary 1. *Let either $n \geq 2$, $\mu_n = \mu_n^-$, and $p(t) \leq 0$ for $t \in [a,b]$ or $n \geq 3$, $\mu_n = \mu_n^+$, and $p(t) \geq 0$ for $t \in [a,b]$. Let, moreover, α, $\beta \in]a,b[$ exist such that $a < \alpha < \beta < b$ and*

$$
\int_\alpha^\beta |p(t)|dt \geq \mu_n\Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}.
$$

Then Eq. (1) *is conjugate in $[a,b]$.*

Note that $\max\{l!(n-l)! : l \in \{1,\dots,n-1\}\} = (n-1)!$. Thus from Theorem 1 easily follows

Corollary 2. *Let $l \in \{1, \dots, n-1\}$, let the conditions* (4) *be fulfilled, and let* $\alpha, \beta \in]a, b[$ *exist such that* $a < \alpha < \beta < b$ *and*

$$\int_\alpha^\beta |p(t)|dt \geq (n-1)!\Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}. \tag{14}$$

Then Eq. (1) *is* $(l, n-l)$ *conjugate in* $[a, b]$.

Note that in the inequality (14) the factor $(n-1)!$ cannot be replaced by $(n-1)! - \varepsilon$ with $\varepsilon \in]0, 1[$. This is shown by the following

Example. Let $\varepsilon \in]0, 1[$ be given beforehand and choose $\alpha \in]a, b[$ and $\beta \in]\alpha, b[$ such that

$$(n-1)!\Big(\frac{\alpha-a}{\beta-a}\Big)^{n-1}\Big(\frac{b-\beta}{b-a}\Big)^{n-1} > (n-1)! - \varepsilon. \tag{15}$$

Put

$$v(t) = \begin{cases} t-a & \text{for} \quad t \in [a,\, \alpha], \\ \dfrac{\alpha+\beta}{2} - a - \dfrac{1}{2(\beta-\alpha)}(t-\beta)^2 & \text{for} \quad t \in]\alpha,\, \beta[, \\ \dfrac{\alpha+\beta}{2} - a & \text{for} \quad t \in [\beta,\, b], \end{cases}$$

$$u_0(t) = \frac{1}{(n-3)!}\int_a^t (t-s)^{n-3} v(s)ds \qquad \text{for} \quad t \in [a, b],\ n \geq 3,$$

$$u_0(t) = v(t) \quad \text{for} \quad t \in [a, b],\ n = 2,$$

and

$$p(t) = \frac{v''(t)}{u_0(t)} \quad \text{for} \quad a < t < b.$$

Then the function u_0 is non-decreasing for $t \in [a, \beta]$, and the inequality

$$u_0(t) \leq u_0(\beta) \leq \frac{1}{(n-3)!}\int_a^\beta (\beta-s)^{n-3}(s-a)ds = \frac{(\beta-a)^{n-1}}{(n-1)!}$$

is valid. Taking into account the inequality (15), we obtain

$$\int_\alpha^\beta p(t)dt = \frac{1}{\beta-\alpha}\int_\alpha^\beta \frac{dt}{u_0(t)} > \frac{(n-1)!}{(\beta-a)^{n-1}} >$$
$$> ((n-1)! - \varepsilon)\Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}.$$

On the other hand, in the case considered, Eq. (1) is not conjugate in $[a, b]$ because it has a solution u_0 satisfying the following conditions:

$$u_0^{(i)}(a) = 0\ (i = 0, \dots, n-2),\ u_0^{(n-1)}(a) = 1,\ u_0(t) > 0 \ \text{ for } a < t \leq b.$$

This example shows that in Corollary 2 inequality (14) cannot be replaced by the inequality

$$\int_{\alpha}^{\beta} |p(t)|dt \geq \big((n-1)! - \varepsilon\big)\Big(\frac{b-a}{(b-\beta)(\alpha-a)}\Big)^{n-1}$$

no matter how small $\varepsilon > 0$ is.

Now consider Eq. (1) on the whole axis $\mathbb{R}$ with $p \in L_{loc}(\mathbb{R})$. From Corollary 2 easily follows

Corollary 3. *Let $l \in \{1, \dots, n-1\}$, p is not zero on the set of the positive measure and*

$$n \geq 2 + \frac{1+(-1)^{n-l}}{2}, \quad (-1)^{n-l}p(t) \geq 0 \quad \textit{for} \quad t \in \mathbb{R}.$$

Then Eq. (1) *is* $(l, n-l)$ *conjugate in* $\mathbb{R}$.

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(Received 1.07.1992)[1]

[1]The original manuscript was prepared for publication by D. Paatashvili