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ON INTEGRAL BERNSTEIN OPERATORS IN SOME CLASSES OF MEASURABLE BIVARIATE FUNCTIONS

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Abstract. The two main theorems are cencerned with the approximation of (complex-valued) functions on the real plane by sums of Bernstein pseudoentire functions. They are formulated and proved in Section 4, after prior determination of the suitable integral operators. Analogous results for pseudopolynomial approximations were obtained by Brudnyĭ, Gonska, and Jetter ([2],[3]).

1. Preliminaries. Let $L_{loc}(R)$ [resp. $AC_{loc}(R)$] be the set of all univariate (complex-valued) functions Lebesgue-integrable (absolutely continuous) on every compact subinterval of $R := (-\infty, \infty)$. Denote by $L^p_{loc}(R^2)$, $1 \le p \le \infty$, the set of all measurable bivariate (complex-valued) functions Lebesgue-integrable with pth power (essentially bounded when $p = \infty$) on every finite two-dimensional integral lying on the plane $R^2 = R \times R$; write $L_{loc}(R^2)$ instead of $L^1_{loc}(R^2)$. Denote by $C(R)$ [resp. $C(R^2)$] the set of all (complex-valued) functions continuous on $R\,[R^2]$.

Given any bivariate (complex-valued) function $f \equiv f(\cdot,\cdot)$ measurable on R^2, the quantity

$$\|f\|_p := \begin{cases} \left(\iint_{R^2} |f(u,v)|^p du\, dv\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \text{ess sup}_{(u,v)\in R^2} |f(u,v)| & \text{if } p = \infty \end{cases} \tag{1}$$

is finite or infinite. In the case $\|f\|_p < \infty$ the function f is said to be of class $L^p \equiv L^p(R^2)$; in symbols, $f(\cdot,\cdot) \in L^p$. The notations $f(\cdot,\cdot) \in L^p_{loc}(R^2)$, $g(\cdot, v) \in L_{loc}(R)$, etc. have a similar meaning.

Let $f \equiv f(\cdot,\cdot)$ be a (complex-valued) function defined on R^2 and let $k, l \in N_0 \equiv \{0, 1, 2, \dots\}$. Determine the partial differences of f at a point

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$(x, y) \in R^2$, i.e.,

$$^1\Delta_\lambda^k f(x,y) := \sum_{\mu=0}^{k} (-1)^{k-\mu} \binom{k}{\mu} f(x+\mu\lambda, y),$$

$$^1\Delta_\eta^l f(x,y) := \sum_{\nu=0}^{k} (-1)^{l-\nu} \binom{l}{\nu} f(x, y+\nu\eta),$$

with the real increments λ, η. Introduce also the mixed difference $\Delta_{\lambda,\eta}^{k,l} := {}^1\Delta_\lambda^k\bigl({}^2\Delta_\eta^l f(x,y)\bigr)$.

It is easy to see that

$$\Delta_{\lambda,\eta}^{k,l} = \sum_{\mu=0}^{k}\sum_{\nu=0}^{l} (-1)^{k+l-\mu-\nu} \binom{k}{\mu}\binom{l}{\nu} f(x+\mu\lambda, y+\nu\eta).$$

In particular, $\Delta_{\lambda,\eta}^{0,0} f(x,y) = f(x,y)$ and

$$\Delta_{\lambda,\eta}^{1,1} f(x,y) = f(x+\lambda, y+\eta) - f(x+\lambda, y) - f(x, y+\eta) + f(x,y).$$

Further, $\Delta_{\lambda,\eta}^{1,1}\bigl(\Delta_{\lambda,\eta}^{k,l} f(x,y)\bigr) = \Delta_{\lambda,\eta}^{k+1,l+1} f(x,y)$.

The first (weak) derivative of f at (x,y) is given by

$$f^{(1)}(x,y) := \lim_{\lambda\to 0} \frac{1}{\lambda^2} \Delta_{\lambda,\lambda}^{1,1} f(x,y)$$

whenever the right-hand side exists. The (weak) derivatives of f of higher orders are defined successively:

$$f^{(j)}(x,y) := \bigl(f^{(j-1)}\bigr)^{(1)}(x,y) \quad \text{for } j = 2, 3, \ldots.$$

Moreover, by convention, $f^{(0)}(x,y) \equiv f(x,y)$ and

$$f^{(m,n)}(x,y) \equiv \frac{\partial^m}{\partial x^m}\Bigl(\frac{\partial^n}{\partial y^n} f(x,y)\Bigr) \quad \text{for } m, n \in N_0.$$

Considering any (complex-valued) function f measurable on R^2, one can define its mixed L^p-modulus of smoothness:

$$\omega_{k,l}(\delta_1, \delta_2; f)_p := \sup\bigl\{\|\Delta_{\lambda,\eta}^{k,l} f\|_p : 0 \le \lambda \le \delta_1,\ 0 \le \eta \le \delta_2\bigr\}.$$

This quantity, with fixed $p \ge 1$ and $k, l \in N_0$, may be finite or infinite for positive numbers δ_1, δ_2. If there exist three non-negative numbers M, α, β such that $\omega_{k,l}(\delta_1, \delta_2; f)_p \le M\delta_1^\alpha \delta_2^\beta$ for all $\delta_1, \delta_2 \in (0, 1]$, we say that f belongs to the Hölder class $H_{\alpha,\beta,p}^{(k,l)}$. More generally, if

$$\|f\|_{\varphi,\psi,p}^{(k,l)} \equiv \sup\Bigl\{\frac{1}{\varphi(\lambda)\psi(\eta)}\|\Delta_{\lambda,\eta}^{k,l} f\|_p : 0 < \lambda, \eta \le 1\Bigr\} < \infty, \tag{2}$$

where φ, ψ mean positive non-decreasing functions on $(0,1]$ and $\varphi(1) = \psi(1) = 1$, the function f is said to be of class $H^{(k,l)}_{\varphi,\psi,p}$. In the case $\varphi(\delta) = \delta^\alpha$, $\psi(\delta) = \delta^\beta$ for $\delta \in (0,1]$, the left-hand side of identity (2) will be denoted by $\|f\|^{(k,l)}_{\alpha,\beta,p}$.

Denote by E_σ [resp. $E_{\sigma,\tau}$] the set of all univariate (bivariate) entire functions of exponential type of order σ [(σ,τ)] at most. Clearly, if $F(\cdot,\cdot) \in E_{\sigma,\tau}(\sigma,\tau \geq 0)$, then $F(\cdot,v) \in E_\sigma$ and $F(u,\cdot) \in E_\tau$ for all $u,v \in R$. Moreover, $F(\cdot,\cdot) \in C(R^2)$. In the case where $\Phi(\cdot,v) \in E_\sigma$ (resp. $\Psi(u,\cdot) \in E_\tau$) for almost every $v \in R$ [$u \in R$], $\Phi(z,\cdot) \in L_{loc}(R)$ [$\Psi(\cdot,z) \in L_{loc}(R^2)$] for every complex number z, and $\Phi(\cdot,\cdot) \in L_{loc}(R^2)$ [$\Psi(\cdot,\cdot) \in L_{loc}(R^2)$], we call Φ [Ψ] a pseudoentire function of class W^1_σ [W^2_τ].

The aim of this paper is to present the Jackson type theorems, in L^p-norms (1) and seminorms (2), for some (complex-valued) functions defined and measurable on R^2. We begin with auxiliary results about the mixed differences and Bernstein's singular integrals used in our approximation problems.

2. Estimates for the mixed differences and moduli of smoothness. Consider a (complex-valued) function $f \equiv f(\cdot,\cdot)$ defined and measurable on the plane R^2. Denote by k,l two non-negative integers. Take an arbitrary p satisfying the condition: $1 \leq p \leq \infty$.

Proposition 1. *If $\lambda,\eta \in R$ and $n \in N \equiv \{1,2,3,\dots\}$, then*

$$\|\Delta^{k,l}_{m\lambda,n\eta} f\|_p \leq m^k n^l \|\Delta^{k,l}_{\lambda,\eta} f\|_p. \tag{3}$$

Proof. Given arbitrary $x,y \in R$, let $g(x,y) := {}^2\Delta^l_{n\eta} f(x,y)$.

By identity (5) of [5], p.116,

$$g(x,y) = \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_l=0}^{n-1} {}^2\Delta^l_\eta f(x, y+\nu_1\eta+\cdots+\nu_i\eta)$$

and

$${}^1\Delta^k_{m\lambda} g(x,y) = \sum_{\mu_1=0}^{m-1} \cdots \sum_{\mu_k=0}^{m-1} {}^1\Delta^k_\lambda g(x+\mu_1\lambda+\cdots+\mu_k\lambda, y).$$

Hence

$${}^1\Delta^{k,l}_{m\lambda,n\eta} f(x,y) = \sum_{\mu_1=0}^{m-1} \cdots \sum_{\mu_k=0}^{m-1} \sum_{\nu_1=0}^{n-1} \cdots$$

$$\cdots \sum_{\nu_l=0}^{n-1} \Delta^{k,l}_{\lambda,\eta} f(x+\mu_1\lambda+\cdots+\mu_k\lambda, y+\nu_1\eta+\cdots+\nu_i\eta).$$

Applying Minkowski's inequality, we get at once estimate (3). $\square$

From (3) it follows that

$$\omega_{k,l}(m\delta_1, n\delta_2; f)_p \le m^k n^l \omega_{k,l}(\delta_1, \delta_2; f)_p \quad (m, n \in N) \tag{4}$$

for all non-negative numbers δ_1, δ_2, whence , in case $a, b \ge 0$

$$\omega_{k,l}(a\delta_1, b\delta_2; f)_p \le ([a]+1)^k([b]+1)^l \omega_{k,l}(\delta_1, \delta_2; f)_p. \tag{5}$$

Clearly, the estimates (3)-(5) are useful only with the finite right-hand sides.

Proposition 2. *Let f have the partial derivatives*

$$f^{(0,n)}(u,\cdot), \dots, f^{(m-1,n)}(u,\cdot) \in L_{loc}(R) \quad (m, n \in N)$$

for every $u \in R$, and let $f^{(m-1,n)}(\cdot, v) \in AC_{loc}(R)$ for almost every $v \in R$. Further, given any $c > 0$, suppose the existence of positive number M_c such that ess $\sup_{-c \le u \le c} |f^{(m,n)}(u,v)| \le M_c$ *for almost every $v \in [-c, c]$. Then, in the case $k \ge m$ and $\lambda, \eta \in R$,*

$$\|\Delta^{k,l}_{\lambda,\eta} f\|_p \le |\lambda|^m |\eta|^n \|\Delta^{k-m,l-n}_{\lambda,\eta} f^{(m,n)}\|_p. \tag{6}$$

Proof. By our assumption, the (Lebesgue) integrals

$$\int_0^\eta \cdots \int_0^\eta f^{(j,n)}(u, w + t_1 + \cdots + t_n) dt_1 \cdots dt_n \equiv F_j(u,w)$$
$$(j = 0, 1, \dots, m)$$

exist for all $u, w \in R$ if $j < m$ and for almost every u and every $w \in R$ if $j = m$.

Given $u, w \in R$, one has

$$\Big|\frac{1}{h}\int_0^h f^{(m,n)}(u+s, w + t_1 + \cdots + t_n) ds\Big| \le M_c$$
$$\text{uniformly in } \; h \in [-1, 1]$$

($h \ne 0$) for almost every point $(t_1, \cdots, t_n)$ of the n-dimensional interval $[-|\eta|, |\eta|]^n$ whenever $c \ge \max(|u| + 1, |w| + n|\eta|)$. Moreover,

$$\lim_{h\to 0} \frac{1}{h}\int_0^h f^{(m,n)}(u+s, w + t_1 + \cdots + t_n) ds =$$
$$= f^{(m,n)}(u, w + t_1 + \cdots + t_n)$$

for almost every u. Hence

$$\lim_{h\to 0} \frac{1}{h}\{F_{m-1}(u+h, w) - F_{m-1}(u,w)\} =$$
$$= \int_0^\eta \cdots \int_0^\eta f^{(m,n)}(u, w + t_1 + \cdots + t_n) dt_1 \cdots dt_n,$$

i.e., $F^{(1,0)}_{m-1}(u,w) = F_m(u,w)$ for a.e. $u \in R$ and every $w \in R$, by the Lebesgue dominated convergence theorem.

Next, when $m \geq 2$, $(u,w) \in R^2$ and $0 < |h| \leq 1$,

$$\frac{1}{h}\{F_{m-2}(u+h,w) - F_{m-2}(u,w)\} =$$
$$= \int_0^\eta \cdots \int_0^\eta \Big\{\frac{1}{h}\int_0^h f^{(m-1,n)}(u+s, w+t_1+\cdots+t_n)ds\Big\}dt_1\cdots dt_n,$$
$$\Big|\frac{1}{h} f^{(m-1,n)}(u+s,w+t_1+\cdots+t_n)ds\Big| =$$
$$= \Big|\frac{1}{h}\Big\{\int_0^s f^{(m,n)}(u+z,w+t_1+\cdots+t_n)dz +$$
$$+ f^{(m-1,n)}(u,w+t_1+\cdots+t_n)\Big\}ds\Big| \leq$$
$$\leq \Big|\frac{1}{h}|f^{(m,n)}(u+z,w+t_1+\cdots+t_n)|dz\Big| +$$
$$+|f^{(m-1,n)}(u,w+t_1+\cdots+t_n)|$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f^{(m-1,n)}(u+s,w+t_1+\cdots+t_n)ds =$$
$$= f^{(m-1,n)}(u,w+t_1+\cdots+t_n)$$

for almost all $(t_1,\ldots,t_n) \in [-|\eta|,|\eta|]^n$. Therefore, as previously, $F^{(1,0)}_{m-2}(u,w) = F_{m-1}(u,w)$ for all $(u,w) \in R^2$ Analogously, when $m \geq 3$, $F^{(1,0)}_{m-3}(u,w) = F_{m-2}(u,w)$ for all $(u,w) \in R^2$, etc. Consequently, if $m \geq 1$, $F^{(m-1,0)}_0(u,w) = F_{m-1}(u,w)$ for all $(u,w) \in R^2$, and $F^{(m,0)}_0(u,w) = F_m(u,w)$ for a.e. $u \in R$ and every $w \in R$

Further, in the case $-c \leq a \leq b \leq c$ and $|w| + n|\eta| \leq c$,

$$|F_{m-1}(b,w) - F_{m-1}(a,w)| =$$
$$= \Big|\int_0^\eta \cdots \int_0^\eta \Big\{\int_a^b f^{(m,n)}(s,w+t_1+\cdots+t_n)ds\Big\}dt_1\cdots dt_n \leq$$
$$\leq M_c(b-a)|\eta|^n.$$

Hence $F_{m-1}(\cdot,w) \in AC_{loc}(R)$ for every $w \in R$.

By identity (4) of [5], p.116,

$$\Delta_{\lambda,\eta}^{m,n} f(x,y) = {}^1\Delta_\lambda^m\bigl({}^2\Delta_\eta^n f(x,y)\bigr) = {}^1\Delta_\lambda^m F_0(x,y) =$$
$$= \int_0^\lambda \cdots \int_0^\lambda F_0^{(m,0)}(x+s_1+\cdots+s_m, y)ds_1\cdots ds_m =$$
$$= \int_0^\lambda \cdots \int_0^\lambda F_m(x+s_1+\cdots+s_m, y)ds_1\cdots ds_m$$

for arbitrary $x, y, \lambda, \eta \in R$. Thus

$$\Delta_{\lambda,\eta}^{m,n} f(x,y) = \int_0^\lambda \cdots \int_0^\lambda \Bigl\{ \int_0^\eta \cdots$$
$$\cdots \int_0^\eta f^{(m,n)}(x+s_1+\cdots+s_m, y+t_1+\cdots+t_n)dt_1...dt_n\Bigr\}ds_1...ds_m.$$

Observing that

$$\Delta_{\lambda,\eta}^{k,l} f(x,y) = \Delta_{\lambda,\eta}^{k-m,l-n}\bigl(\Delta_{\lambda,\eta}^{m,n} f(x,y)\bigr) =$$
$$= \int_0^\lambda \cdots \int_0^\lambda \Bigl\{ \int_0^\eta \cdots \int_0^\eta \Delta_{\lambda,\eta}^{k-m,l-n} \times$$
$$\times f^{(m,n)}(x+s_1+\cdots+s_m, y+t_1+\cdots+t_n)dt_1\cdots dt_n\Bigr\}ds_1\cdots ds_m$$

and applying the generalized Minkowski's inequality, we obtain (6). $\square$

Estimate (6) immediately implies

$$\omega_{k,l}(\delta_1,\delta_2;f)_p \le \delta_1^m\delta_2^n\omega_{k-m,l-n}(\delta_1,\delta_2;f^{(m,n)})_p \quad \text{for} \quad \delta_1,\delta_2 \ge 0. \qquad (7)$$

Proposition 3. *Let $(f_0, f_1, \ldots, f_\rho)$ be a system of (complex-valued) functions of two real variables, such that $f_\rho \in L_{loc}(R^2)$ $(\rho \in N)$ and, for $j = 1, \ldots, \rho$,*

$$f_{\rho-j}(x,y) = f_{\rho-j}(x,0) + f_{\rho-j}(0,y) - f_{\rho-j}(0,0) +$$
$$+ \int_0^x \int_0^y f_{\rho-j+1}(u,v)\,du\,dv \quad \textit{if}\; x,y \ne 0,$$

$f_{\rho-j}(x,0)$, $f_{\rho-j}(0,y)$ are defined for all real x, y and, when $j < \rho$, $f_{\rho-j}(\cdot,0)$, $f_{\rho-j}(0,\cdot) \in L_{loc}(R)$. Suppose that the integer k is greater than or equal to ρ. Then, for all $\lambda, \eta \in R$

$$\|\Delta_{\lambda,\eta}^{k,k} f_0\|_p \le |\lambda\eta|^\rho \|\Delta_{\lambda,\eta}^{k-\rho,k-\rho} f_\rho\|_p. \qquad (8)$$

Proof. Given arbitrary $x, y, \lambda, \eta \in R$, we have

$$\Delta^{1,1}_{\lambda,\eta} f_0(x,y) = \int_x^{x+y} \int_y^{y+\eta} f_1(s,t)\, ds\, dt =$$

$$= \int_0^\lambda \int_0^\eta f_1(x+s_1, y+t_1)\, ds_1 dt_1,$$

$$\Delta^{2,2}_{\lambda,\eta} f_0(x,y) = \int_0^\lambda \int_0^\eta \Delta^{1,1}_{\lambda,\eta} f_1(x+s_1, y+t_1)\, ds_1 dt_1 =$$

$$= \int_0^\lambda \int_0^\eta \Big\{ \int_0^\lambda \int_0^\eta f_2(x+s_1+s_2, y+t_1+t_2) ds_2 dt_2 \Big\}\, ds_1 dt_1, \quad \text{etc.}$$

Therefore

$$\Delta^{k,k}_{\lambda,\eta} f_0(x,y) = \Delta^{k-\rho,k-\rho}_{\lambda,\eta} \big(\Delta^{\rho,\rho}_{\lambda,\eta} f_0(x,y) \big) =$$

$$= \int_0^\lambda \cdots \int_0^\lambda \int_0^\eta \cdots \int_0^\eta \Delta^{k-\rho,k-\rho}_{\lambda,\eta} \times$$

$$\times f_\rho(x+s_1+\cdots+s_\rho, y+t_1+\cdots+t_\rho) ds_1 \cdots ds_\rho dt_1 \cdots dt_\rho.$$

This immediately implies estimate (8). $\square$

From (8) it follows that

$$\omega_{k,k}(\delta_1, \delta_2; f_0)_p \le (\delta_1 \delta_2)^\rho \omega_{k-\rho,k-\rho}(\delta_1, \delta_2; f_\rho)_p \tag{9}$$

for all non-negative numbers δ_1, δ_2.

3. Basic properties of the Bernstein singular integrals. Consider the entire functions g_σ, $G_{\sigma,k}$ of exponential type of positive order σ, with positive integer parameters r, k given by

$$g_\sigma(z) := \Big(\frac{1}{z} \sin \frac{\sigma z}{2r} \Big)^{2r}, \quad G_{\sigma,k}(\zeta) := \sum_{\mu=1}^{k} (-1)^\mu \frac{1}{\mu} \binom{k}{\mu} g_\sigma\big(\frac{\zeta}{\mu}\big).$$

Write

$$\gamma_\sigma := \int_R g_\sigma(t)\, dt = 2 \Big(\frac{\sigma}{2r} \Big)^{2r-1} \int_0^\infty \Big(\frac{\sin v}{v} \Big)^{2r} dv.$$

Suppose that $f \equiv f(\cdot,\cdot)$ is a (complex-valued) function defined and measurable on R^2, such that

$$I(f) \equiv \iint_{R^2} \frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}\, du\, dv < \infty. \tag{10}$$

Take arbitrary $\tau > 0$, $l \in N$ and complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ $(x_m, y_m \in R)$. Introduce the singular integrals

$$J_{\sigma,\tau}[f](z_1,z_2) :=(\gamma_\sigma\gamma_\tau)^{-1}\iint_{R^2} f(u,v)G_{\sigma,k}(z_1-u)G_{\tau,l}(z_2-v)\,du\,dv, \quad (11)$$

$$J^1_\sigma[f](z_1,v) := \gamma_\sigma^{-1}\int_R f(u,v)G_{\sigma,k}(z_1-u))\,du \quad (v \in R), \qquad (12)$$

$$J^2_\tau[f](u,z_2) := \gamma_\tau^{-1}\int_R f(u,v)G_{\tau,l}(z_2-v)\,dv \quad (u \in R), \qquad (13)$$

which are due to S.Bernstein ([1], pp.421-432).

The double integral (11) exists in the Lebesgue sense for all complex z_1, z_2; the single (Lebesgue) integrals (12) and (13) exist for every complex z_1 [resp. z_2] and almost every real v [resp. u]. More precise assertions will be presented below.

Proposition 4. *The relation*

$$J_{\sigma,\tau}[f](z_1,z_2) = O\big((1+x_1^{2r})(1+x_2^{2r})e^{\sigma|y_1|+\tau|y_2|}I(f)\big) \qquad (14)$$

holds uniformly in $x_1, y_1, x_2, y_2 \in R$; moreover, $J_{\sigma,\tau}[f]$ coincides with some bivariate entire function.

Proof. Putting

$$S(z_1,z_2) := \iint_{R^2} f(u,v)g_\sigma(z_1-u)g_\tau(z_2-v)\,du\,dv,$$

we can, formally, write

$$S(z_1,z_2) = \Big(\int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1} + \int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty} +$$
$$+\int_{x_1+1}^{\infty}\int_{x_2-1}^{x_2+1} + \cdots\Big) f(u,v)g_\sigma(z_1-u)g_\tau(z_2-v)\,du\,dv =$$
$$= T_0(z_1,z_2) + T_1(z_1,z_2) + T_2(z_1,z_2) + \cdots .$$

In view of (10), $f \in L_{loc})R^2)$. Hence the term $T_0(z_1,z_2)$ exists because $g_\sigma \in E_\sigma$, $g_\tau \in E_\tau$. Furthermore,

$$|T_0(z_1,z_2)| \le \int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1} |f(u,v)|\Big(\frac{\sigma}{2r}\Big)^{2r} e^{\sigma|y_1|}\Big(\frac{\tau}{2r}\Big)^{2r} e^{\tau|y_2|}du\,dv \le$$
$$\le \Big(\frac{\sigma\tau}{4r^2}\Big)^{2r} e^{\sigma|y_1|+\tau|y_2|}\int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1}\frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}du\,dv\,\cdot$$
$$\cdot\big\{1+(|x_1|+1)^{2r}\big\}\big\{1+(|x_2|+1)^{2r}\big\} \le$$
$$\le 4\Big(\frac{\sigma\tau}{4r^2}\Big)^{2r} e^{\sigma|y_1|+\tau|y_2|}I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r}.$$

Next

$$|T_1(z_1,z_2)| \le \int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty} |f(u,v)| e^{\sigma|y_1|}(x_1-u)^{-2r} e^{\tau|y_2|}(x_2-v)^{-2r}\,du\,dv \le$$

$$\le e^{\sigma|y_1|+\tau|y_2|}\int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty}\frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}du\,dv\,\cdot$$

$$\cdot\{1+(|x_1|+1)^{2r}\}\{1+(|x_2|+1)^{2r}\} \le$$

$$\le 4e^{\sigma|y_1|+\tau|y_2|} I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r},$$

$$|T_2(z_1,z_2)| \le \int_{x_1+1}^{\infty}\int_{x_2-1}^{\infty} |f(u,v)| e^{\sigma|y_1|}(x_1-u)^{-2r}\Big(\frac{\tau}{2r}\Big)^{2r} e^{\tau|y_2|}du\,dv \le$$

$$\le 4\Big(\frac{\tau}{2r}\Big)^{2r} e^{\sigma|y_1|+\tau|y_2|} I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r}, \text{ etc.}$$

Thus

$$S(z_1,z_2) = O\big((1+x_1^{2r})(1+x_2^{2r})e^{\sigma|y_1|+\tau|y_2|} I(f)\big) \tag{15}$$

uniformly in $x_1, y_1, x_2, y_2 \in R$.

Evidently, the left-hand side of (15) can be replaced by

$$S_{\mu,\nu}(z_1,z_2) := \iint_{R^2} f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big) g_\tau\Big(\frac{z_2-v}{\nu}\Big) du\,dv \quad (\mu,\nu \in N).$$

Further,

$$J_{\sigma,\tau}[f](z_1,z_2) = \frac{1}{\gamma_\sigma\gamma_\tau}\sum_{\mu=1}^{k}\sum_{\nu=1}^{l}(-1)^{\mu+\nu}\binom{k}{m}\binom{l}{\nu}\frac{1}{\mu\nu}S_{\mu,\nu}(z_1,z_2).$$

Therefore, the uniform relation (14) is established. Applying the Lebesgue dominated convergence theorem, it can easily be proved that the function $J_{\sigma,\tau}[f]$ is continuous at every point belonging to the space of pairs of complex numbers.

The (Lebesgue) integrals

$$\int_{-n}^{n}\int_{-n}^{n} f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big) g_\tau\Big(\frac{z_2-v}{\nu}\Big) du\,dv \quad (\mu,\nu,n \in N)$$

define some entire functions $F_{\mu,\nu,n}$ of two complex variables z_1, z_1, because they have the partial derivatives

$$\frac{\partial F_{\mu,\nu,n}}{\partial z_1}(z_1,z_2) = \frac{1}{\mu}\int_{-n}^{n}\int_{-n}^{n} f(u,v) g'_\sigma\Big(\frac{z_1-u}{\mu}\Big) g_\tau\Big(\frac{z_2-v}{\nu}\Big) du\,dv$$

and

$$\frac{\partial F_{\mu,\nu,n}}{\partial z_2}(z_1,z_2) = \frac{1}{\nu}\int_{-n}^{n}\int_{-n}^{n} f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big) g'_\tau\Big(\frac{z_2-v}{\nu}\Big) du\,dv$$

An easy calculation shows that $\lim_{n\to\infty} F_{\mu,\nu,n}(z_1,z_2) = S_{\mu,\nu}(z_1,z_2)$ uniformly in z_1, z_2 belonging to two arbitrary bounded sets of complex numbers. Hence the well-known Weierstrass theorem ensures that all $S_{\mu,\nu}$ are entire functions of two variables. Consequently, $J_{\sigma,\tau}[f]$ is a bivariate entire function for every fixed pair (σ,τ) of positive numbers. Obviously, in view of (14), $J_{\sigma,\tau}[f](\cdot,\cdot) \in E_{\sigma,\tau}$. □

Proposition 5. *The singular integral* (12), *with a positive parameter* σ, *has the following basic properties:*
(i) $J^1_\sigma[f](\cdot,v) \in E_\sigma$ *for almost every* $v \in R$,
(ii) $J^1_\sigma[f](z_1,\cdot) \in L_{loc}(R)$ *for every complex* z_1,
(iii) $J^1_\sigma[f](\cdot,\cdot) \in L_{loc}(R^2)$.

Proof of (i). By Fubini's theorem, condition (10) implies

$$\Big|\int_{-\infty}^{\infty} \frac{f(u,v)}{1+u^{2r}} du\Big| < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{|f(u,v)|}{1+u^{2r}} du < \infty$$

for almost every $v \in R$. Hence, for these v and $z_1 = x_1 + iy_1$,

$$\varphi_\mu(z_1,v) \equiv \int_{-\infty}^{\infty} \Big|f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big)\Big| du < \infty \ \ (\mu = 1,\dots,k)$$

and

$$Y(z_1,v) \equiv \int_R f(u,v) G_{\sigma,k}(z_1-u) du =$$
$$= \sum_{\mu=1}^{k} (-1)^\mu \frac{1}{\mu} \binom{k}{\mu} \int_{-\infty}^{\infty} f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big) du.$$

Writing

$$\varphi_\mu(z_1,v) = \Big(\int_{x_1-1}^{x_1+1} + \int_{x_1+1}^{\infty} + \int_{-\infty}^{x_1-1}\Big) \Big|f(u,v) g_\sigma\Big(\frac{z_1-u}{\mu}\Big)\Big| du$$

and proceeding as in the proof of Proposition 4, we obtain

$$\varphi_\mu(z_1,v) = O\Big((1+x_1^{2r}) e^{(\sigma/\mu)|y_1|} \int_R |f(u,v)|(1+u^{2r})^{-1} du\Big) \tag{16}$$

uniformly in x_1, y_1, v. This immediately implies the O-relation for $Y(z_1,v)$, in which the right side is as in (16) with $\mu = 1$.

Further, $Y(\cdot,v)$ are entire functions, by the Weierstrass theorem. Consequently, $Y(\cdot,v) \in E_\sigma$ for almost every v. Thus the assertion (i) is obtained. □

Proof of (ii). Given $\mu, n \in N$ and any complex number $z_1 = x_1 + iy_1$, we have

$$\int_{-\infty}^{\infty}\int_{-n}^{n}\Big|f(u,v)g_\sigma\Big(\frac{z_1-u}{\mu}\Big)\Big|du\,dv \le$$
$$\le \int_{x_1-1}^{x_1+1}\int_{-n}^{n}|f(u,v)|\Big(\frac{\sigma}{2r}\Big)^{2r}e^{\sigma|y_1|/\mu}du\,dv +$$
$$+\Big(\int_{-\infty}^{x_1-1}\int_{-n}^{n}+\int_{x_1+1}^{\infty}\int_{-n}^{n}\Big)|f(u,v)|\Big(\frac{\mu}{x_1-u}\Big)^{2r}e^{\sigma|y_1|/\mu}du\,dv < \infty.$$

Applying Fubini's theorem we conclude that $Y(z_1,\cdot) \in L_{loc}(R)$, which implies (ii). □

Proof of (iii). Let $\mu, n \in N$, $x_1 \in R$. By Tonelli's theorem,

$$\int_{-\infty}^{\infty}\int_{-n}^{n}\int_{-n}^{n}\Big|f(u,v)g_\sigma\Big(\frac{x_1-u}{\mu}\Big)\Big|du\,dv\,dx_1 =$$
$$= \int_{-n}^{n}\Big\{\int_{-\infty}^{\infty}\int_{-n}^{n}|f(u,v)|g_\sigma\Big(\frac{x_1-u}{\mu}\Big)du\,dv\Big\}dx_1. \tag{17}$$

The inner double integral of the right-hand side of (17) does not exceed

$$\int_{x_1-1}^{x_1+1}\int_{-n}^{n}|f(u,v)|\Big(\frac{\sigma}{2r}\Big)^{2r}du\,dv +$$
$$+\Big(\int_{-\infty}^{x_1-1}\int_{-n}^{n}+\int_{x_1+1}^{\infty}\int_{-n}^{n}\Big)|f(u,v)|\Big(\frac{\mu}{x_1-u}\Big)^{2r}du\,dv \le$$
$$\le \Big(\frac{\sigma}{2r}\Big)^{2r}\int_{-n-1}^{n+1}\int_{-n}^{n}|f(u,v)|du\,dv + 2\mu^{2r}(1+n)^{2r})(1+n^{2r})I(f).$$

Thus the left-hand side of (17) is finite.

Now, the Fubini theorem ensures that $Y(\cdot,\cdot) \in L_{loc}(R^2)$, and (iii) follows. □

Analogous properties of the singular integral (13) can easily be formulated. They will also be used in the sequel.

4. Approximations by the sums of pseudoentire functions. Let f and the operators $J_{\sigma,\tau}$, J_σ^1, J_τ^2 be as in Section 3. Putting

$$\Phi(z_1,v) := -J_{\sigma,\tau}[f](z_1,v) - J_\sigma^1[f](z_1,v),$$
$$\Psi(u,z_2) := J_\tau^2[f](u,z_2),$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ $(x_j, y_j \in R)$, and $u, v \in R$, it can easily be observed (see Propositions 4,5) that $\Phi(\cdot,\cdot) \in W_\sigma^1$, $\Psi(\cdot,\cdot) \in W_\tau^2$.

Introduce the approximant $Q_{\sigma,\tau}[f](x_1,x_2) := \Phi(x_1,x_2)+\Psi(x_1,x_2)$, which is defined almost everywhere on R^2 and $Q_{\sigma,\tau}[f] \in L_{loc}(R^2)$. Assuming that

$k,l \leq 2r-2$ $(k,l,r \in N)$ and $1 \leq p \leq \infty$, we will present some Jackson type estimates.

In discussions below, the symbols $C_j(q,\dots)$ will mean positive constants depending on the indicated parameters $q,\dots$, only.

Theorem 1. *Under the restriction* $0<\sigma,\tau<\infty$, *we have*

$$\|f-Q_{\sigma,\tau}[f]\|_p \leq 2^{k+l}C_1(r)\omega_{k,l}(1/\sigma,1/\tau;f)_p.$$

Proof. Take into account the real numbers x_1,x_2 for which $J^1_\sigma[f](\cdot,x_2)\in E_\sigma$, $J^2_\tau[f](x_1,\cdot)\in E_\tau$. In this case,

$$f(x_1,x_2)-Q_{\sigma,\tau}[f](x_1,x_2)= \\ =\frac{(-1)^{k+l}}{\gamma_\sigma\gamma_\tau}\iint_{R^2}\Delta^{k,l}_{s,t}f(x_1,x_2)g_\sigma(s)g_\tau(t)\,ds\,dt. \tag{18}$$

Hence, by Minkowski's inequality and (5),

$$\gamma_\sigma\gamma_\tau\|f-Q_{\sigma,\tau}[f]\|_p \leq \\ \leq \omega_{k,l}(1/\sigma,1/\tau;f)_p\iint_{R^2}(\sigma|s|+1)^k(\tau|t|+1)^l g_\sigma(s)g_\tau(t)\,ds\,dt.$$

Further,

$$\frac{1}{\gamma_\sigma}\int_R(\sigma|s|+1)^k g_\sigma(s)\,ds \leq \frac{2}{\gamma_\sigma}\Big\{2^k\int_0^{1/\sigma}g_\sigma(s)\,ds+ \\ +2^k\sigma^k\int_{1/\sigma}^\infty s^k g_\sigma(s)\,ds\Big\} \leq 2^k\Big\{1+\frac{2}{\gamma_\sigma}\sigma^k\int_{1/\sigma}^\infty s^{k-2r}ds\Big\} \leq \\ \leq 2^k\Big\{1+\frac{2}{\gamma_\sigma}\sigma^{2r-1}\Big\}.$$

Observing that

$$\gamma_\sigma \geq 2\Big(\frac{\sigma}{2r}\Big)^{2r-1}\int_0^{\pi/2}\Big(\frac{2}{\pi}\Big)^{2r}dv = 2\Big(\frac{\sigma}{\pi r}\Big)^{2r-1},$$

we obtain

$$\frac{1}{\gamma_\sigma}\int_R(\sigma|s|+1)^k g_\sigma(s)\,ds \leq 2^k\{1+(\pi r)^{2r-1}\}.$$

Thus

$$\|f-Q_{\sigma,\tau}[f]\|_p \leq \omega_{k,l}(1/\sigma,1/\tau;f)_p\cdot 2^{k+l}\{1+(\pi r)^{2r-1}\}^2,$$

and the proof is complete. □

Corollary 1. *Let f satisfy all conditions of Proposition 2, with some positive integers $m \le k$, $n \le l$. Then*

$$\|f - Q_{\sigma,\tau}[f]\|_p \le 2^{k+l} C_1(r) \sigma^{-m} \tau^{-n} \omega_{k-m,l-n}(1/\sigma, 1/\tau; f^{(m,n)})_p$$

This estimate is an immediate consequence of Theorem 1 and inequality (7).

Corollary 2. *Consider the bivariate functions $f_0, f_1, \dots, f_\rho$ defined in Proposition 3, with $f_0(\cdot,0)$, $f_0(0,\cdot) \in L_{loc}(R)$ and $f_{\rho-j}(\cdot,0)$, $f_{\rho-j}(0,\cdot) \in C(R)$ when $1 \le j \le \rho - 1$. Suppose that for some non-negative numbers $a, b < 2r - \rho - 1$ and for $j = 1, \dots, \rho$ the relations*

$$f_\rho(x,y) = O\big((1+|x|^a)(1+|y|^b)\big),$$
$$f_{\rho-j}(x,0) = O\big(1+|x|^{a+j}\big), \quad f_{\rho-j}(0,y) = O\big(1+|y|^{b+j}\big)$$

hold uniformly in $x, y \in R$. Then if $k = l \ge \rho$, the function $f = f_0$ and its approximant $Q_{\sigma,\tau}[f]$ have weak derivatives of order $\rho - 1$ everywhere on R^2 and

$$\|f^{(\mu)} - Q^{(\mu)}_{\sigma,\tau}[f]\|_p \le C_2(k,r)(\sigma\tau)^{\mu-\rho} \omega_{k-\rho+\mu,k-\rho+\mu}\Big(\frac{1}{\sigma}, \frac{1}{\tau}; f_\rho\Big)_p$$

for $\mu = 0, \dots, \rho - 1$. Under the additional assumption $f_\rho \in C(R^2)$, also the derivatives $f^{(\rho)}$ and $Q^{(\rho)}_{\sigma,\tau}[f]$ exist on R^2 and the last inequality remains valid for $\mu = \rho$.

Indeed, an easy calculation shows that

$$f^{(\mu)}(x,y) = f_\mu(x,y) \quad \text{and} \quad Q^{(\mu)}_{\sigma,\tau}[f](x,y) = Q_{\sigma,\tau}[f_\mu](x,y)$$

for all $(x,y) \in R^2$, whenever $0 \le \mu \le \rho - 1$ or $\mu = \rho$ and $f_\rho \in C(R^2)$. By Theorem 1 and a suitable estimate analogous to (9), we obtain

$$\|f_\mu - Q_{\sigma,\tau}[f]\|_p \le C_2(k,r)\omega_{k,k}(1/\sigma, 1/\tau; f_\mu)_p \le$$
$$\le C_2(k,r)(\sigma\tau)^{\mu-\rho}\omega_{k-\rho+\mu,k-\rho+\mu}(1/\sigma, 1/\tau; f_\rho)_p \ \text{ if } 0 \le \mu \le \rho.$$

The conclusion is now evident.

Theorem 2. *Let $f \in H^{(k,l)}_{\varphi,\psi,p}$, where φ, ψ are as in Section 1. Denote by α and β two non-negative numbers such that $t^{-\alpha}\varphi(t)$, $t^{-\beta}\psi(t)$ are non-decreasing on $(0,1]$. Then, for $\sigma, \tau \ge 1$,*

$$\|f - Q_{\sigma,\tau}[f]\|^{(k,l)}_{\alpha,\beta,p} \le C_3(k,l,r)\|f\|^{(k,l)}_{\varphi,\psi,p}\big\{\sigma^\alpha \varphi(1/\sigma) + \tau^\beta \psi(1/\tau)\big\}.$$

Proof. Write $D_{\sigma,\tau}(x,y) := f(x,y) - Q_{\sigma,\tau}[f](x,y)$ $(x,y \in R)$ In view of (18),

$$\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}(x,y) = \frac{1}{\gamma_\sigma \gamma_\tau} \times$$
$$\times \iint_{R^2} g_\sigma(s) g_\tau(t) \sum_{\mu=0}^{k} \sum_{\nu=0}^{l} (-1)^{\mu+\nu} \binom{k}{\mu} \binom{l}{\nu} \Delta^{k,l}_{s,t} f(x+\mu\lambda, y+\nu\eta) ds\, dt$$

for all $\lambda, \eta \in R$ and almost all $(x,y) \in R^2$; whence, by Minkowski's inequality,

$$\|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\| \le \frac{2^{k+l}}{\gamma_\sigma \gamma_\tau} \iint_{R^2} g_\sigma(s) g_\tau(t) \|\Delta^{k,l}_{s,t} f\|_p ds\, dt.$$

Proceeding now as in the proof of Theorem 1, we obtain

$$\|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\|_p \le 4^{k+l} C_1(r) \|f\|^{(k,l)}_{\varphi,\psi,p} \varphi(1/\sigma) \psi(1/\tau). \tag{19}$$

On the other hand, equality (18) leads to

$$\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}(x,y) = \frac{1}{\gamma_\sigma \gamma_\tau} \times$$
$$\times \iint_{R^2} g_\sigma(s) g_\tau(t) \sum_{\mu=0}^{k} \sum_{\nu=0}^{l} (-1)^{\mu+\nu} \binom{k}{\mu} \binom{l}{\nu} \Delta^{k,l}_{\lambda,\eta} f(x+\mu s, y+\nu t) ds\, dt$$

for almost all $(x,y) \in R^2$, which implies

$$\|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\|_p \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \varphi(\lambda) \psi(\eta) \quad \text{when} \quad \lambda, \eta \in (0,1]. \tag{20}$$

Next, $\|D_{\sigma,\tau}\|^{(k,l)}_{\alpha,\beta,p} = \sup\{\Omega(\lambda,\eta) : 0 < \lambda, \eta \le 1\}$, where $\Omega(\lambda,\eta) = \lambda^{-\alpha}\eta^{-\beta} \|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\|_p$. If $1/\sigma \le \lambda \le 1$ and $1/\tau \le \eta \le 1$ then, by (19), $\Omega(\lambda,\eta) \le 4^{k+l} C_1(r) \|f\|^{(k,l)}_{\varphi,\psi,p} \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau)$. From (20) it follows that

$$\Omega(\lambda,\eta) \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \times$$
$$\times \begin{cases} \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) & \text{if } 0 < \lambda \le 1/\sigma,\ 0 < \eta \le 1/\tau, \\ \sigma^\alpha \varphi(1/\sigma) & \text{if } 0 < \lambda \le 1/\sigma,\ 1/\tau \le \eta \le 1, \\ \tau^\beta \psi(1/\tau) & \text{if } 1/\sigma \le \lambda \le 1,\ 0 < \eta \le 1/\tau. \end{cases}$$

Hence

$$\|D_{\sigma,\tau}\|^{(k,l)}_{\alpha,\beta,p} \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \{2^{k+l} C_1(r) \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) +$$
$$+ \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) + \sigma^\alpha \varphi(1/\sigma) + \tau^\beta \psi(1/\tau)\},$$

and the proof is completed (cf. [4], Theorem 2). $\square$

Corollary 3. *If* $\varphi(\delta) = \delta^{\alpha'}$, $\psi(\delta) = \delta^{\beta'}$ *for all* $\delta \in (0,1]$ *then, in the case* $\alpha' \geq \alpha \geq 0$ *and* $\beta' \geq \beta \geq 0$, $\|f - Q_{\sigma,\tau}[f]\|_{\alpha,\beta,p}^{(k,l)} = O\big(\sigma^{\alpha-\alpha'} + \tau^{\beta-\beta'}\big)$ *uniformly in* $\sigma, \tau \geq 1$.

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