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ON CONVERGENCE SUBSYSTEMS OF ORTHONORMAL SYSTEMS

G. BARELADZE

Abstract. It is proved that for any sequence $\{R_k\}_{k=1}^{\infty}$ of real numbers satisfying $R_k \geq k$ $(k \geq 1)$ and $R_k = o(k \log_2 k)$, $k \to \infty$, there exists a orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0;1)$, such that none of its subsystems $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ with $n_k \leq R_k$ $(k \geq 1)$ is a convergence subsystem.

Let $\{\varphi_n(x)\}$ be an orthonormal system (ONS) on $(0;1)$. It is called a convergence system if the series $\sum c_n \varphi_n(x)$ is convergent almost everywhere whenever the sequence $\{c_n\}$ of real numbers satisfies $\sum c_n^2 < \infty$.

It is well-known [1] that not every ONS $\{\varphi_n(x)\}$ is a convergence system. However [2], [3], each of them contains some convergence subsystem $\{\varphi_{n_k}(x)\}$. A question was formulated later [4] whether there exists a common estimate of growth rate of numbers n_k in the class of all ONS. B.S.Kashin [5] answered this question in the affirmative: one can determine a sequence of positive numbers $\{R_k\}$ such that from any ONS it is possible to choose a convergence subsystem $\{\varphi_{n_k}\}$ with $n_k \leq R_k$, $1 \leq k < \infty$. In the same paper [5] the problem of finding $\{R_k\}$ with a minimal admissible growth order is formulated and the hypothesis $R_k = k^{1+\varepsilon}$ $(\varepsilon > 0)$ is conjectured. G.A.Karagulyan [6] proved that one can take $R_k = \lambda^k$, $\lambda > 1$. However, this upper estimate is rougher than the one expected in [5].

In this paper we shall give the proof of the theorem providing the lower estimate for $\{R_k\}$.

Theorem. *For any sequence $\{R_k\}_{k=1}^{\infty}$ of real numbers satisfying*

$$R_k \geq k \ \ (k \geq 1) \quad \text{and} \quad R_k = o(k \log_2 k), \ \ k \to \infty, \tag{1}$$

there exists an ONS $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0;1)$, such that none of its subsystems $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ with $n_k \leq R_k$ $(k \geq 1)$ is a convergence subsystem.

Several lemmas are needed to prove this theorem.

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Lemma 1 (H.Rademacher [7]). *For any ONS* $\{\psi_n(x)\}_{n=1}^N$, $x\in(0;1)$, *and any collection of real numbers* $\{c_n\}_{n=1}^N$

$$\int_0^1 \Big(\max_{1\le j\le N}\Big|\sum_{n=1}^{j} c_n\psi_n(x)\Big|\Big)^2 dx \le c\log_2^2(N+1)\sum_{n=1}^{N} c_n^2, \quad 1\le N<\infty.^1$$

Lemma 2. *For any* $N\ge 1$ *there exists an ONS*

$$\psi(N) := \{\psi_n^N(x)\}_{n=1}^N, \quad x\in(0;1),$$

satisfying, for any collection of natural numbers $1\le n_1<n_2<\cdots<n_m\le N$ $(1\le m\le N)$, *the inequality*

$$\int_0^1 \max_{1\le j\le m}\Big|\sum_{k=1}^{j}\psi_{n_k}^N(x)\Big|dx \ge c\frac{\log_2 N}{\sqrt{N}}m. \tag{2}$$

Proof. We shall assure that the requirements of this lemma are satisfied by the ONS usually used in the proof of the Menshov-Rademacher theorem. The functions $\psi_n^N(x)$, $1\le n\le N$, belonging to this ONS (see [8], p.295) have, in particular, the following properties:

(i)

$$\psi_n^N(x) := \begin{cases} \frac{c\sqrt{N}}{s-n}, & x\in\left(\frac{s-1}{2N};\frac{2s-1}{4N}\right),\\ \frac{c\sqrt{N}}{n-s}, & x\in\left(\frac{2s-1}{4N};\frac{s}{2N}\right), \quad 1\le s\le N, \quad s\ne n;\\ 0, & x\in\left(\frac{n-1}{2N};\frac{n}{2N}\right);\end{cases}$$

(ii) $\psi_n^N(x)$ is constant on each of the intervals

$$\Big(\frac{s-1}{4N};\frac{s}{4N}\Big), \quad 2N+1\le s\le 4N;$$

(iii)

$$\int_0^1 \psi_n^N(x)\,dx = 0;$$

(iv) $\psi_n^N(x)$ is extended from $(0;1)$ onto $(-\infty;\infty)$ with period 1.

Denote $\delta_s := \left(\frac{s-1}{2N};\frac{2s-1}{4N}\right)$, $1\le s\le N$. When $x\in\delta_s$, because of (i) we have $\psi_p^N(x)\ge 0$ for $1\le p\le s$ and $\psi_p^N(x)\le 0$ for $s\le p\le N$ $(1\le s\le N)$.

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[1] Here and in what follows c denotes positive absolute constants which, in general, may differ from one equality (inequality) to another.

Therefore for fixed numbers $1 \le n_1 < n_2 < \cdots < n_m \le N$ $(1 \le m \le N)$ and for each $x \in \delta_s$ $(1 \le s \le N)$ we obtain

$$\sum_{k=1}^{m} \left|\psi_{n_k}^N(x)\right| = \sum_{k:n_k \le s} + \sum_{k:n_k > s} = \Big| \sum_{k:n_k \le s} \psi_{n_k}^N(x) \Big| +$$

$$+ \Big| \sum_{k:n_k > s} \psi_{n_k}^N(x) \Big| \le 3 \max_{1 \le j \le m} \Big| \sum_{k=1}^{j} \psi_{n_k}^N(x) \Big|.$$

Hence

$$\int_{\delta_s} \max_{1 \le j \le m} \Big| \sum_{k=1}^{j} \psi_{n_k}^N(x) \Big| dx \ge \frac{1}{3} \int_{\delta_s} \sum_{k=1}^{m} \left|\psi_{n_k}^N(x)\right| dx =$$

$$= \frac{c}{\sqrt{N}} \sum_{\substack{k:1 \le k \le m, \\ n_k \ne s}} \frac{1}{|n_k - s|}, \quad 1 \le s \le N;$$

$$\int_0^1 \max_{1 \le j \le m} \Big| \sum_{k=1}^{j} \psi_{n_k}^N(x) \Big| dx \ge \sum_{s=1}^{N} \int_{\delta_s} \max_{1 \le j \le m} \Big| \sum_{k=1}^{j} \psi_{n_k}^N(x) \Big| dx \ge$$

$$\ge \frac{c}{\sqrt{N}} \sum_{s=1}^{N} \sum_{\substack{k:1 \le k \le m, \\ n_k \ne s}} \frac{1}{|n_k - s|} =$$

$$= \frac{c}{\sqrt{N}} \sum_{k=1}^{m} \sum_{\substack{s:1 \le s \le N, \\ s \ne n_k}} \frac{1}{|n_k - s|} \ge c \frac{\log_2 N}{\sqrt{N}} m. \ \blacksquare$$

Remark 1. For any positive integer Q, $\{\psi_n^N(Qx)\}_{n=1}^N$ is an ONS on $(0;1)$ also satisfying the inequality (2).

Remark 2. Let N_0, N_1, Q_0, Q_1, p be positive integers. If $Q_1 = 4pN_0Q_0$, then functions belonging to different collections $\{\psi_n^{N_0}(Q_0x)\}_{n=1}^{N_0}$ and $\{\psi_n^{N_1}(Q_1x)\}_{n=1}^{N_1}$ are mutually orthogonal and pairwise stochastically independent on $(0;1)$.

Both conclusions follow readily from (i)–(iv).

Lemma 3. *Suppose a function $f \in L^1_{(0;1)}$ is not equivalent to zero and*

$$A := \big\{x \in (0;1) : |f(x)| > \frac{1}{2}\|f\|_{L^1_{(0;1)}}\big\}.$$

Then

$$\operatorname{mes} A \ge \|f\|^2_{L^1_{(0;1)}} / 4\|f\|^2_{L^2_{(0;1)}}. \tag{3}$$

Proof. Indeed, we obtain (using Hölder's inequality)

$$\|f\|_{L^1_{(0;1)}} = \int_A |f(x)|dx + \int_{(0;1)\setminus A} |f(x)|dx \leq$$
$$\leq (\operatorname{mes} A)^{1/2} \cdot \|f\|_{L^2_{(0;1)}} + \frac{1}{2}\|f\|_{L^1_{(0;1)}},$$

which immediately implies (3). □

Lemma 4. *Let N, Q, m $(1 \leq m \leq N)$ be positive integers. Then for any collection of natural numbers $1 \leq n_1 < n_2 < \cdots < n_m \leq N$*

$$\operatorname{mes}\Big\{x \in (0;1) \,:\, \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j}\psi_{n_k}^N(Qx)\Big| > \frac{J}{2}\Big\} \geq c\frac{m}{N},$$

where $\psi_n^N(x) \in \psi(N)$, $1 \leq n \leq N$ (see Lemma 2), and

$$J := \int_0^1 \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j}\psi_{n_k}^N(Qx)\Big| dx.$$

Proof. By Lemmas 1 and 2 (see also Remark 1) we have

$$\int_0^1 \Big(\max_{1\leq j\leq m}\Big|\sum_{k=1}^{j}\psi_{n_k}^N(Qx)\Big|\Big)^2 dx \leq cm\log_2^2(m+1),$$
$$J^2 \geq c\frac{m^2}{N}\log_2^2 N; \quad 1 \leq m \leq N < \infty.$$

Thus, applying Lemma 3 , we obtain

$$\operatorname{mes}\Big\{x \in (0;1) \,:\, \max_{1\leq j\leq m}\Big|\sum_{k=1}^{j}\psi_{n_k}^N(Qx)\Big| > \frac{J}{2}\Big\} \geq c\frac{m}{N}\cdot\frac{\log_2^2 N}{\log_2^2(m+1)} \geq c\frac{m}{N}. \ \blacksquare$$

Proof of the theorem. If the conditions (1) are fulfilled, then $R_k = k\log_2 k/\varepsilon(k)$, $2 \leq k < \infty$, where $\varepsilon(k)$ tends to infinity. Moreover, without loss of generality, it can be assumed that

a) $R_1 = 1$;
b) $\varepsilon(k)$ is a nondecreasing sequence of positive integers;
c) the sets $\Delta_m := \{k : \varepsilon(k) = m\}$ have the form $(\nu_{m-1};\nu_m] \cap \mathbb{N}$ $(m = 1, 2, \dots)$, where $\nu_0 = 0$, $\log_2\log_2\log_2 \nu_m = p_m$ $(m \geq 1)$ and $\{p_m\}_{m=1}^{\infty}$ is some increasing sequence of positive integers.

In particular, for $k \in \Delta_m$ $(m \geq 2)$ we have

$$\begin{aligned}\log_2 k > \log_2 \nu_{m-1} = 2^{2^{p_{m-1}}} \geq 2^{2^{m-1}} > m = \varepsilon(k),\\ \varepsilon(k\log_2 k) \leq \varepsilon(\nu_m\log_2 \nu_m) < \varepsilon(\nu_{m+1}) = m+1 \leq 2\varepsilon(k).\end{aligned} \tag{4}$$

Denote

$$T_1 := 0, \quad T_k := 2^{[k\log_2 k \log_2\log_2 k]}, \quad 2 \le k < \infty, \tag{5}$$

and $E_m := \{k : R_k \le T_m\}$, $2 \le m < \infty$ (here $[x]$ is the integer part of the number x).

Since the function $\varphi(x) = x\varepsilon(x)/\log_2 x$ increases on $(e;\infty)$, taking into account (4) and (5), we have, for $m > \nu_1$,

$$\begin{aligned} E_m &= \{1;2\} \cup \big\{k \ge 3 : R_k \le T_m\big\} = \{1;2\} \cup \Big\{k \ge 3 : \frac{k\log_2 k}{\varepsilon(k)} \le T_m\Big\} = \\ &= \{1;2\} \cup \Big\{k \ge 3 : \varphi\Big(\frac{k\log_2 k}{\varepsilon(k)}\Big) \le \varphi(T_m)\Big\} \supset \\ &\supset \{1;2\} \cup \big\{k \ge 3 : 2k \le \varphi(T_m)\big\} = \Big\{k \ge 1 : k \le \frac{1}{2}\varphi(T_m)\Big\}. \end{aligned}$$

Therefore for a large m ($m \ge m_1 > \nu_1$)

$$|E_m| > \frac{1}{2}\varphi(T_m) - 1 \ge \frac{1}{3}\,\frac{T_m\varepsilon(T_m)}{\log_2 T_m}, \tag{6}$$

$$|E_{m+1}| \ge \frac{1}{3}\,\frac{T_{m+1}\varepsilon(T_{m+1})}{\log_2 T_{m+1}} \ge \frac{1}{3}\,\frac{T_{m+1}}{T_m\log_2 T_{m+1}} \cdot T_m \ge 2T_m; \tag{7}$$

Because of

$$\sum_{k\in\Delta_m} \frac{1}{\log_2 T_k} = \sum_{k\in\Delta_m} \frac{1}{[k\log_2 k\log_2\log_2 k]} > \frac{1}{8}(p_m - p_{m-1}) \ge \frac{1}{8} \quad (m \ge 2)$$

we can select a subsequence $T_{q_k} \equiv \widetilde{T}_k$ ($1 \le k < \infty$; $\widetilde{T}_1 = 0$, $\widetilde{T}_2 = \nu_1$) such that

$$\frac{1}{2m^2} < \sum_{k:q_k\in\Delta_m} \frac{1}{\log_2 \widetilde{T}_k} \equiv \sum_{k:q_k\in\Delta_m} \frac{1}{\log_2 T_{q_k}} \le \frac{1}{m^2}, \quad m \ge 2, \tag{8}$$

and hence

$$\begin{aligned} \sum_{k:q_k\in\Delta_m} \frac{\varepsilon(\widetilde{T}_k)}{\log_2 \widetilde{T}_k} &= \sum_{k:q_k\in\Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} \ge \\ \ge \sum_{k:q_k\in\Delta_m} \frac{\varepsilon(q_k)}{\log_2 T_{q_k}} &= m \sum_{k:q_k\in\Delta_m} \frac{1}{\log_2 T_{q_k}} > \frac{1}{2m}, \quad m \ge 2. \end{aligned} \tag{9}$$

Let

$$\begin{gathered} N_m := \widetilde{T}_m - \widetilde{T}_{m-1}, \\ Q_2 := 1, \quad Q_{m+1} := 4N_mQ_m, \quad 2 \le m < \infty. \end{gathered} \tag{10}$$

Consider the orthonormal collections

$$\{\psi_n^{N_m}(Q_m x)\}_{n=1}^{N_m}, \quad 2 \le m < \infty,$$

and construct with their aid the desired ONS $\{\varphi_n(x)\}_{n=1}^{\infty}$ as

$$\varphi_n(x) := \psi_k^{N_m}(Q_m x), \tag{11}$$

where $n \in (\widetilde{T}_{m-1}; \widetilde{T}_m]$, $k = n - \widetilde{T}_{m-1}$, $2 \le m < \infty$, $x \in (0;1)$ (the orthonormality follows from (10) and Remark 2).

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers with $k \le n_k \le R_k$, $1 \le k < \infty$. We set

$$\begin{aligned}
&G_m := \{k : \widetilde{T}_{m-1} < n_k \le \widetilde{T}_m\},\\
&M_m := |G_m|,\\
&a_n := \big((1+M_m)\log_2 \widetilde{T}_m\big)^{-1/2} \ \text{ for } \ n \in (\widetilde{T}_{m-1}; \widetilde{T}_m]; \ \ m \ge 2.
\end{aligned}$$

On account of (8)

$$\sum_{k=1}^{\infty} a_{n_k}^2 = \sum_{m=2}^{\infty} \sum_{k:\widetilde{T}_{m-1} < n_k \le \widetilde{T}_m} a_{n_k}^2 = \sum_{m=2}^{\infty} \frac{M_m}{(1+M_m)\log_2 \widetilde{T}_m} <$$
$$< \sum_{k=2}^{\infty} \frac{1}{\log_2 T_{q_k}} = \sum_{m=2}^{\infty} \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} < \infty.$$

It is thus sufficient to show that the series

$$\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}(x) \tag{12}$$

diverges on some set of positive measure.

Note that

$$G_m = \{k : n_k \le \widetilde{T}_m\} \backslash \{k : n_k \le \widetilde{T}_{m-1}\} \supset$$
$$\supset \{k : R_k \le \widetilde{T}_m\} \backslash \{k : k \le \widetilde{T}_{m-1}\}, \quad m \ge 2.$$

Consequently in view of (6) and (7)

$$M_m \ge |\{k : R_k \le \widetilde{T}_m\}| - |\{k : k \le \widetilde{T}_{m-1}\}| = |\{k : R_k \le T_{q_m}\}| - T_{q_{m-1}} >$$
$$> |E_{q_m}| - \frac{1}{2}|E_{q_{m-1}+1}| \ge \frac{1}{2}|E_{q_m}| \ge \frac{1}{6}\,\frac{\widetilde{T}_m \varepsilon(\widetilde{T}_m)}{\log_2 \widetilde{T}_m}, \quad m \ge m_1. \tag{13}$$

Hence by (11) and Lemma 2

$$\begin{aligned}\widetilde{J}_m := \int_0^1 \max_{\widetilde{T}_{m-1}<j\le\widetilde{T}_m} \Big| \sum_{k:\widetilde{T}_{m-1}<n_k\le j} a_{n_k}\varphi_{n_k}(x)\Big| dx \ge \\ \ge c\cdot \frac{1}{\sqrt{(1+M_m)\log_2 \widetilde{T}_m}}\cdot \frac{\log_2 N_m}{\sqrt{N_m}}\cdot M_m \ge \\ \ge c\sqrt{\frac{M_m \log_2 \widetilde{T}_m}{\widetilde{T}_m}} \ge c\sqrt{\varepsilon(\widetilde{T}_m)}, \quad m\ge m_1,\end{aligned}$$

and therefore

$$\lim_{m\to\infty} \widetilde{J}_m = \infty. \tag{14}$$

If A_m denotes the set

$$\Big\{x\in(0;1): \max_{\widetilde{T}_{m-1}<j\le\widetilde{T}_m} \Big| \sum_{k:\widetilde{T}_{m-1}<n_k\le j} a_{n_k}\varphi_{n_k}(x)\Big| > \frac{1}{2}\widetilde{J}_m\Big\}, \quad m\ge 2,$$

then by (11), Lemma 4, (13) and (9)

$$\begin{aligned}\operatorname{mes} A_m \ge \frac{M_m}{N_m} > \frac{M_m}{\widetilde{T}_m} > \frac{1}{6}\,\frac{\varepsilon(\widetilde{T}_m)}{\log_2 \widetilde{T}_m}, \quad m\ge m_1; \\ \sum_{m=2}^{\infty} \operatorname{mes} A_m \ge \frac{1}{6}\sum_{k=m_1}^{\infty} \frac{\varepsilon(\widetilde{T}_k)}{\log_2 \widetilde{T}_k} = \frac{1}{6}\sum_{k:q_k\ge q_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} > \\ > \frac{1}{6}\sum_{k:q_k>\nu_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \frac{1}{6}\sum_{m=1+m_1}\;\sum_{k:q_k\in\Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \infty.\end{aligned} \tag{15}$$

It is easy to verify (see (10),(11), Remark 2) that $\{A_m\}_{m=2}^{\infty}$ is a sequence of stochastically independent sets. Therefore by (15) and the Borel-Cantelli lemma

$$\operatorname{mes}\big(\lim_{m\to\infty}\sup A_m\big) = 1.$$

Hence we conclude because of (14) and the definition of sets A_m that the series (12) diverges almost everywhere on (0;1). ■

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Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., 380043 Tbilisi
Republic of Georgia