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TWO-WEIGHTED L_p-INEQUALITIES FOR SINGULAR INTEGRAL OPERATORS ON HEISENBERG GROUPS

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Abstract. Some sufficient conditions are found for a pair of weight functions, providing the validity of two-weighted inequalities for singular integrals defined on Heisenberg groups.

Estimates for singular integrals of the Calderon–Zygmund type in various spaces (including weighted spaces and the anisotropic case) have attracted a great deal of attention on the part of researchers. In this paper we will deal with singular integral operators T on the Heisenberg group H^n which have an essentially different character as compared with operators of the Calderon–Zygmund type. We have obtained the two-weighted L_p-inequality with monotone weights for singular integral operators T on H^n. Applications are given.

Let H^n be the Heisenberg group (see [1], [2]) realized as a set of points $x=(x_0,x_1,\dots,x_{2n})=(x_0,x')\in\mathbb{R}^{2n+1}$ with the multiplication

$$xy=\Big(x_0+y_0+\frac{1}{2}\sum_{i=1}^{n}(x_iy_{n+i}-x_{n+i}y_i),\quad x'+y'\Big).$$

The corresponding Lie algebra is generated by the left-invariant vector fields

$$X_0=\frac{\partial}{\partial x_0},\quad X_i=\frac{\partial}{\partial x_i}+\frac{1}{2}x_{n+i}\frac{\partial}{\partial x_0},$$
$$X_{n+i}=\frac{\partial}{\partial x_{n+i}}-\frac{1}{2}x_i\frac{\partial}{\partial x_0},\quad i=1,\dots,n,$$

which satisfy the commutation relation

$$[X_i,X_{n+i}]=\frac{1}{4}X_0,$$
$$[X_0,X_i]=[X_0,X_{n+i}]=[X_i,X_j]=[X_{n+i},X_{n+j}]=[X_i,X_{n+j}]=0,$$

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$$i, j = 1, \dots, n \quad i \neq j.$$

The dilation $\delta_t : \delta_t x = (t^2 x_0, tx')$, $t > 0$, is defined on H^n. The Haar measure on this group coincides with the Lebesgue measure $dx = dx_0 dx_1 \cdots dx_{2n}$. The identity element in H^n is $e = 0 \in \mathbb{R}^{2n+1}$, while the element x^{-1} inverse to x is $(-x)$.

The function f defined in H^n is said to be H-homogeneous of degree m, on H^n, if $f(\delta_t x) = t^m f(x)$, $t > 0$. We also define the norm on H^n

$$|x|_H = \big[x_0^2 + \big(\sum_{i=1}^{2n} x_i^2\big)^2\big]^{1/4}$$

which is H-homogeneous of degree one. This also yields the distance function, namely, the distance

$$d(x, y) = d(y^{-1}x, e) = |y^{-1}x|_H,$$

$$|y^{-1}x|_H = \big[\big(x_0 - y_0 - \frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)\big)^2 + \\ + \big(\sum_{i=1}^{2n}(x_i - y_i)^2\big)^2\big]^{1/4}.$$

d is left-invariant in the sense that $d(x, y)$ remains unchanged when x and y are both left-translated by some fixed vector in H^n. Furthermore, d satisfies the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$, $x, y, z \in H^n$. For $r > 0$ and $x \in H^n$ let

$$B(x,r) = \{y \in H^n;\ |y^{-1}x|_H < r\} \quad (S(x,r) = \{y \in H^n;\ |y^{-1}x|_H = r\})$$

be the H-ball (H-sphere) with center x and radius r.

The number $Q = 2n + 2$ is called the homogeneous dimension of H^n. Clearly, $d(\delta_t x) = t^Q dx$.

Given functions $f(x)$ and $g(x)$ defined in H^n, the Heisenberg convolution (H-convolution) is obtained by

$$(f * g)(x) = \int_{H^n} f(y) g(y^{-1}x) dy = \int_{H^n} f(xy^{-1}) g(y) dy,$$

where dy is the Haar measure on H^n.

The kernel $K(x)$ admitting the estimate $|K(x)| \leq C|x|_H^{\alpha - Q}$ is summable in the neighborhood of e for $\alpha > 0$ and in that case $K * g$ is defined for the function g with bounded support. If however the kernel $K(x)$ has a singularity of order Q at zero, i.e., $|K(x)| \sim |x|_H^{-Q}$ near e, then there arises a singular integral on H^n.

Let $\omega(x)$ be a positive measurable function on H^n. Denote by $L_p(H^n,\omega)$ a set of measurable functions $f(x)$, $x \in H^n$, with the finite norm

$$\|f\|_{L_P(H^n,\omega)} = \Big(\int\limits_{H^n} |f(x)|^p \omega(x) dx \Big)^{1/p}, \quad 1 \le p < \infty.$$

We say that a locally integrable function $\omega : H^n \to (0,\infty)$ satisfies Muckenhoupt's condition $A_p = A_p(H^n)$ (briefly, $\omega \in A_p$), $1 < p < \infty$, if there is a constant $C = C(\omega, p)$ such that for any H-ball $B \subset H^n$

$$\Big(|B|^{-1} \int_B \omega(x) dx \Big) \Big(|B|^{-1} \int_B \omega^{1-p'}(x) dx \Big) \le C, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where the second factor on the left is replaced by $\operatorname{ess\,sup}\{\omega^{-1}(x) : x \in B\}$ if $p = 1$.

Let $K(x)$ be a singular kernel defined on $H^n \backslash \{e\}$ and satisfying the conditions: $K(x)$ is an H-homogeneous function of degree $-Q$, i.e., $K(\delta_t x) = t^{-Q} K(x)$ for any $t > 0$ and $\int_{S_H} K(x) d\sigma(x) = 0$, where $d\sigma(x)$ is a measure element on $S_H = S(e,1)$.

Denote by $\omega_K(\delta)$ the modulus of continuity of the kernel on S_H:

$$\omega_K(\delta) = \sup\{|K(x) - K(y)| : x, y \in S_H, \quad |y^{-1}x|_H \le \delta\}.$$

It is assumed that

$$\int_0^1 \omega_K(t) \frac{dt}{t} < \infty.$$

We consider the singular integral operator T:

$$Tf(x) = \int\limits_{H^n} K(xy^{-1}) f(y) dy =: \lim_{\varepsilon \to 0+} \int\limits_{|xy^{-1}|_H > \varepsilon} K(xy^{-1}) f(y) dy.$$

As is known, T acts boundedly in $L_p(H^n)$, $1 < p < \infty$ (see [3], [4]). For singular integrals with Cauchy–Szegö kernels the weighted estimates were established in the norms of $L_p(H^n,\omega)$ with weights ω satisfying the condition A_p [5]. These results extend to the more general kernels considered above [4].

Theorem 1 [4]. *Let $1 < p < \infty$ and $\omega \in A_p$; then T is bounded in $L_p(H^n,\omega)$.*

In the sequel we will use

Theorem 2. *Let $1 \le p \le q < \infty$ and $U(t)$, $V(t)$ be positive functions on $(0,\infty)$.*

1) *The inequality*

$$\Big(\int_0^\infty U(t)\Big|\int_0^t \varphi(\tau)d\tau\Big|^q dt\Big)^{1/q} \le K_1\Big(\int_0^\infty |\varphi(t)|^p v(t)dt\Big)^{1/p}$$

with the constant K_1 not depending on φ holds iff the condition

$$\sup_{t>0}\Big(\int_t^\infty U(\tau)d\tau\Big)^{p/q}\Big(\int_0^t V(\tau)^{1-p'}d\tau\Big)^{p-1} < \infty$$

is fulfilled;

2) *The inequality*

$$\Big(\int_0^\infty U(t)|\int_t^\infty \varphi(\tau)d\tau|^q dt\Big)^{1/q} \le K_2\Big(\int_0^\infty |\varphi(t)|^p V(t)dt\Big)^{1/p}$$

with the constant K_2 not depending on φ holds iff the condition

$$\sup_{t>0}\Big(\int_0^t U(\tau)d\tau\Big)^{p/q}\Big(\int_t^\infty V(\tau)^{1-p'}d\tau\Big)^{p-1} < \infty$$

is fulfilled.

Note that Theorem 2 was proved by G.Talenti, G.Tomaselli, B.Muckenhoupt [7] for $1 \le p = q < \infty$, and by J.S.Bradley [8], V.M.Kokilashvili [9], V.G.Maz'ya [10] for $p < q$.

We say that the weight pair (ω, ω_1) belongs to the class $\widetilde{A}_{pq}(\gamma)$, $\gamma > 0$, if either of the following conditions is fulfilled: a) $\omega(t)$ and $\omega_1(t)$ are increasing functions on $(0, \infty)$ and

$$\sup_{t>0}\Big(\int_t^\infty \omega(\tau)\tau^{-1-\gamma q/p'}d\tau\Big)^{p/q}\Big(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{\gamma-1}d\tau\Big)^{p-1} < \infty;$$

b) $\omega(t)$ and $\omega_1(t)$ are decreasing functions on $(0, \infty)$ and

$$\sup_{t>0}\Big(\int_0^{t/2} \omega_1(\tau)\tau^{\gamma-1}d\tau\Big)^{p/q}\Big(\int_t^\infty \omega(\tau)^{1-p'}\tau^{-1-\gamma p'/q}d\tau\Big)^{p-1} < \infty.$$

Theorem 3. *Let $1 < p < \infty$ and the weight pair $(\omega, \omega_1) \in \widetilde{A}_p(Q) \equiv \widetilde{A}_{pp}(Q)$. Then for $f \in L_p(H^n, \omega(|x|_H))$ there exists $Tf(x)$ for almost all $x \in H^n$ and*

$$\int_{H^n} |Tf(x)|^p \omega_1(|x|_H)dx \le C\int_{H^n} |f(x)|^p \omega(|x|)_H)dx, \tag{1}$$

where the constant C does not depend on f.

Corollary. *If $\omega(t)$, $t > 0$ is increasing (decreasing) and the function $\omega(t)t^{-\beta}$ is decreasing (increasing) for some $\beta \in (0, Q(p-1))$ ($\beta \in (-Q, 0)$), then T is bounded on $L_p(H^n, \omega(|x|_H))$.*

Proof of Theorem 3. Let $f \in L_p(H^n, \omega(|x|_H))$ and ω, ω_1 be positive increasing functions on $(0, \infty)$. We will prove that $Tf(x)$ exists for almost all $x \in H^n$. We take any fixed $\tau > 0$ and represent the function f in the norm of the sum $f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & \text{if } |x|_H > \tau/2 \\ 0, & \text{if } |x|_H \leq \tau/2 \end{cases}, \quad f_2(x) = f(x) - f_1(x).$$

Let $\omega(t)$ be a positive increasing function on $(0, \infty)$ and $f \in L_p(H^n, \omega(|x|_H))$. Then $f_1 \in L_p(H^n)$ and therefore $Tf_1(x)$ exists for almost all $x \in H^n$. Now we will show that Tf_2 converges absolutely for all $x : |x|_H \geq \tau$. Note that $C(K) = \sup_{x \in S_H} |K(x)| < \infty$. Hence

$$\begin{aligned} |Tf_2(x)| &\leq C(K) \int\limits_{|y|_H \leq \tau/2} \frac{|f(y)|}{|xy_{-1}|_H^Q} dy \leq \\ &\leq \Big(\frac{2}{\tau}\Big)^{\frac{Q}{p}} \int\limits_{|y|_H \leq \tau/2} \frac{|f(y)|\omega(|y|_H)^{\frac{1}{p}}}{\omega(|y|_H)^{\frac{1}{p}}} dy, \end{aligned} \tag{2}$$

since $|xy^{-1}|_H \geq |x|_H - |y|_H \geq \tau/2$. Thus, by the Hölder inequality we can estimate (2) as

$$|Tf_2(x)| \leq C\tau^{-Q/p} \|f\|_{L_p(H^n, \omega(|x|_H))} \Big(\int_0^{\tau/2} \omega(t)^{1-p'} t^{Q-1} dt\Big)^{1/p'}.$$

Therefore $Tf_2(x)$ converges absolutely for all $x : |x|_H \geq \tau$ and thus $Tf(x)$ exists for almost all $x \in H^n$. Assume $\bar{\omega}_1(t)$ to be an arbitrary continuous increasing function on $(0, \infty)$ such that $\bar{\omega}_1(t) \leq \omega_1(t), \bar{\omega}_1(0) = \omega_1(0+)$ and $\bar{\omega}_1(t) = \int_0^t \varphi(\tau) d\tau + \bar{\omega}_1(0)$, $t \in (0, \infty)$ (it is obvious that such $\bar{\omega}_1(t)$ exists; for example, $\bar{\omega}_1(t) = \int_0^t \omega_1'(\tau) d\tau + \omega_1(t)$).

We observe that the condition a) implies

$$\exists C_1 > 0, \ \forall t > 0, \quad \omega_1(t) \leq C_1 \omega(t/2). \tag{3}$$

Indeed, from

$$\begin{gathered} \exists C_2 > 0, \ \forall t > 0, \\ \Big(\int_t^\infty \varphi(\tau) \tau^{-Q(p-1)} d\tau\Big) \Big(\int_0^{t/2} \omega(\tau)^{1-p'} \tau^{Q-1} d\tau\Big)^{p-1} \leq C_2 \end{gathered} \tag{4}$$

we obtain (3), since

$$\int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau \geq C\omega_1(t)t^{-Q(p-1)},$$
$$\Big(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{Q-1}d\tau\Big)^{p-1} \leq C\omega(t/2)^{-1}t^{Q(p-1)}$$

and, besides,

$$\frac{1}{Q(p-1)}\int_t^\infty \varphi(\tau)\tau^{-Q(p-1)}d\tau = \int_t^\infty \varphi(\tau)d\tau \int_\tau^\infty \lambda^{-1-Q(p-1)}d\lambda =$$
$$= \int_t^\infty \lambda^{-1-Q(p-1)}d\lambda \int_t^\lambda \varphi(\tau)d\tau \leq \int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau.$$

We have

$$\|Tf\|_{L_{p,\bar\omega_1(H^n)}} \leq \Big(\int\limits_{H^n} |Tf(x)|^p dx \int_0^{|x|_H} \varphi(t)dt\Big)^{1/p} +$$
$$+\Big(\bar\omega_1(0)\int\limits_{H^n} |Tf(x)|^p dx\Big)^{1/p} = A_1 + A_2.$$

If $\omega(0+) > 0$, then $L_p(H^n, \omega(|x|_H)) \subset L_p(H^n)$, and if $\omega(0+) = 0$, then $\bar\omega(t) \leq \omega_1(t) \leq C\omega(t/2)$ implies $\bar\omega_1(0) = 0$. Therefore in the case $\omega(0+) = 0$ we have $A_2 = 0$.

If $\omega(0) > 0$, then $f \in_{L_p(\mathbb{R}^n)}$ and we have

$$A_2 \leq C\Big(\bar\omega_1(0)\int\limits_{H^n} |f(x)|^p dx\Big)^{1/p} \leq C\Big(\int\limits_{H^n} |f(x)|^p\omega_1(|x|_H)dx\Big)^{1/p} \leq$$
$$\leq C\|f\|_{L_p(H^n,\omega(|x|_H))}.$$

Now we can write

$$A_1 \leq \Big(\int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t} |Tf(x)^p dx\Big)^{1/p} \leq A_{11} + A_{12}.$$

where

$$A_{11}^p = \int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t} \Big|\int\limits_{|y|_H>t/2} K(x,y^{-1})f(y)dy\Big|^p dx,$$
$$A_{12}^p = \int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t} \Big|\int\limits_{|y|_H<t/2} K(x,y^{-1})f(y)dy\Big|^p dx.$$

The relation

$$\int\limits_{|y|_H>t/2} |f(y)|^p dy \le \frac{1}{\omega(t/2)} \int\limits_{|y|_H>t/2} |f(y)|^p \omega(|y|_H) dy$$

implies $f \in L_p(\{y \in H^n : |y|_H > t\})$ for any $t > 0$.

Hence, on account of (3), we have

$$A_{11} \le C\Big(\int_0^\infty \varphi(t)dt \int\limits_{|x|_H>t/2} |f(x)|^p dx\Big)^{1/p} =$$

$$= C\Big(\int\limits_{H^n} |f(x)|^p dx \int_0^{2|x|_H} \varphi(t)dt\Big)^{1/p} \le$$

$$\le C\Big(\int\limits_{H^n} |f(x)|^p \omega_1(2|x|_H)dx\Big)^{1/p} \le C\|f\|_{L_{p,\omega(|x|_H)}}(H^n).$$

Obviously, if $|x|_H > t$, $|y|_H < t/2$, then $\frac{1}{2}|x|_H \le |y^{-1}x|_H \le \frac{3}{2}|x|_H$. Therefore

$$\int\limits_{|x|_H>t} \Big| \int\limits_{|y|_H<t/2} K(xy^{-1})f(y)dy\Big|^p dx \le$$

$$\le C(K) \int\limits_{|x|_H>t} \Big(\int\limits_{|y|_H<t/2} |xy^{-1}|_H^{-Q} |f(y)|dy\Big)^p dx \le$$

$$\le 2^{Qp} C(K) \int\limits_{|x|_H>t} |x|_H^{-Qp} dx \Big(\int\limits_{|y|_H<t/2} |f(y)|dy\Big)^p.$$

Taking the H-polar coordinates $x = \delta_\varrho \bar{x}$, $\varrho = |x|_H$, $\bar{x} \in S_H$ we can write

$$\int\limits_{|x|_H>t} |x|_H^{-Qp} dx = \int\limits_{S_H} d\sigma(\bar{x}) \int_0^\infty \varrho^{Q-1-Qp} d\varrho = Ct^{Q-Qp}.$$

For $\alpha > Q(1+\frac{1}{p'})$, by virtue of the Hölder inequality, we have

$$\int\limits_{|y|_H<t/2} |f(y)|dy = \alpha \int\limits_{S_H} d\sigma(\bar{y}) \int_0^{t/2} \varrho^{Q-\alpha-1} |f(\delta_\varrho \bar{y})| d\varrho \int_0^\varrho s^{\alpha-1} ds =$$

$$= \alpha \int_0^{t/2} s^{\alpha-1} ds \int\limits_{s<|y|_H<t/2} |f(y)|\, |y|_H^{-\alpha} dy \le$$

$$\le \int_0^{t/2} s^{\alpha-1} ds \Big(\int\limits_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} \times$$

$$\times \Big(\int\limits_{s<|y|_H<t/2} |y|_H^{(Q-\alpha)p'} dy \Big)^{1/p'} \le$$

$$\le C \int_0^{t/2} s^{Q+\frac{Q}{p'}} \Big(\int\limits_{s<|y|_H<t/2} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds.$$

Consequently

$$A_{12} \le C \Big\{ \int_0^{\infty} \varphi(2t) t^{-Q(p-1)} \times$$

$$\times \Big[\int_0^t s^{Q(1+\frac{1}{p'})} \Big(\int\limits_{|y|_H \ge s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} ds \Big]^p dt \Big\}^{1/p}.$$

By (4) and Theorem 2

$$A_{12} \le C \Big[\int_0^{\infty} s^{Qp(1+\frac{1}{p'})} \Big(\int\limits_{|y|_H > s} |f(y)|^p \times$$

$$\times |y|^{-Qp} dy \Big) \omega(s) s^{-(Q-1)(p-1)} ds \Big]^{1/p} =$$

$$= C \Big(\int_0^{\infty} s^{-1+Qp} \omega(s) ds \int\limits_{|y|_H > s} |f(y)|^p |y|_H^{-Qp} dy \Big)^{1/p} =$$

$$= C \Big(\int\limits_{H^n} |f(y)|^p |y|_H^{-Qp} \int_0^{|y|_H} \omega(s) s^{-1+Qp} ds \Big)^{1/p} \le$$

$$\le C \Big(\int\limits_{H^n} |f(y)|^p \omega(|y|_H) dy \Big)^{1/p}.$$

Hence we obtain (1) for $\omega_1(t) = \bar{\omega}_1(t)$. Now, by the Fatou theorem, the inequality (1) is fulfilled. □

Theorem 3 was earlier announced in [11].

A similar reasoning can be used to prove the analogue of Theorem 3 for the operator $T_\alpha : f \to T_\alpha f$ where

$$T_\alpha f(x) = \int\limits_{H^n} |xy^{-1}|_H^{\alpha-Q} f(y) dy, \quad 0 < \alpha < Q.$$

Namely, we have

Theorem 4. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ *and the weights* (ω, ω_1) *be monotone positive functions on* $(0, \infty)$. *Then the inequality*

$$\Big(\int\limits_{H^n} |T_\alpha f(x)|^q \omega_1(|x|_H) dx \Big)^{1/q} \leq C \Big(\int\limits_{H^n} |f(x)|^p \omega(|x|_H) dx \Big)^{1/p}$$

holds if and only if $(\omega, \omega_1) \in \widetilde{A}_{p,q}(Q)$.

Remark. In the case of a homogeneous group the analogue of Theorem 4 is also valid (see [12]).

For monotone weights one can find the weighted L_p-estimates for a Calderon–Zygmund operator in [13] and [14], and for the anisotropic case in [15].

As known [16], if $f \in C_0^\infty(H^n)$, then the function

$$g(x) = C_n \int\limits_{H^n} |xy^1|_H^{-2n} f(y) dy$$

is a solution of the equation $L_0 g = f$, where $L_0 = -\sum_{i=1}^{2n} X_j^2$. In particular, our results lead to

Theorem 5. *Let* $1 < p < \infty$, $(\omega, \omega_1) \in \widetilde{A}(Q)$, $f \in L_p(H^n, \omega(|x|_H))$, *and* $L_0(g) = f$. *Then*

$$\|X_0 g\|_{L_p(H^n, \omega_1(|x|_H))} \leq c \|f\|_{L_p(H^n, \omega(|x|_H))},$$
$$\|X_i X_j g\|_{L_p(H^n, \omega_1(|x|)_H))} \leq C \|f\|_{L_p(H^n, \omega(|x|_H))},$$
$$i, j = 1, 2, \dots, 2n.$$

Theorem 6. *Let* $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$, $(\omega, \omega_1) \in \widetilde{A}_{pq}(Q)$, $f \in L_p(H^n, \omega(|x|_H))$, *and* $L_0 g = f$. *Then*

$$\|X_i g\|_{L_q(H^n, \omega_1(|x|_H))} \leq C \|f\|_{L_p(H^n, \omega(|x|_H))}, \quad i = 1, 2, \dots, 2n.$$

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