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PASSAGE OF THE LIMIT THROUGH THE DOUBLE DENJOY INTEGRAL

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Abstract. The conditions are given for passage of the limit through the double Denjoy integral defined by V.G. Chelidze.

As is well-known, there exist various definitions of the double Denjoy integral (see [1,2,5]). Conditions for passage of limits through these integrals have not yet been studied. The object of this paper is to investigate the conditions for passage of the limit through the double Denjoy integral defined by V.G. Chelidze (see [7]).

Here we shall use the well-known terms (see, for example, [8]). We recall only a few definitions.

Definition 1 (see [8], pp. 127-128). A function $f : R_0 \to \mathbb{R}$ with $R_0 = [a,b] \times [c,d]$ is said to be Denjoy integrable (D-integrable) on R_0 or, briefly, $f \in D(R_0)$ if there exists a generalized absolutely continuous function F (ACG-function) on R_0, briefly, $F \in ACG(R_0)$ (see [8], pp. 99-100), with an approximate derivative ([8], p. 103) equal to f a.e. The function F is called a D-primitive of f and $\Delta(f;R_0) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$ is called the D-integral of f on R_0 which is written as

$$\Delta(F,R_0) = (D)\iint\limits_{R_0} f(x,y)dxdy.$$

A function $f : R_0 \to \mathbb{R}$ is called D-integrable on a measurable [1] subset $E \subset R_0$ (briefly $f \in D(E)$), if $f_E(x,y) \in D(R_0)$ where

$$f_E(x,y) = \begin{cases} f(x,y) & (x,y) \in E, \\ 0 & (x,y) \notin E. \end{cases}$$

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[1] Measurability is meant in the Lebesgue sense and $|A|$ will stand for the Lebesgue measure of the set A.

Definition 2. A compact interval I is said to be of type 2 with respect to E if at least one pair of the opposite vertices of I lies in E.

Lemma 1. *Let E_1 be a compact subset of $[a,b]$ whose contiguous intervals are $\{r_i\}_i$. If $f \in D(E)$, $f \in D(\bar{R}_i)$, $i = \overline{1,\infty}$, where $E = E_1 \times [c,d]$, $R_i = r_i \times [c,d]$, and*

$$\sum_{i=1}^{\infty} O(D; f, \bar{R}_i) < \infty, \tag{1}$$

where

$$O[D; f, \bar{R}_i] = \sup_{\rho \subset \bar{R}_i} \left\{ \left| (D) \iint\limits_{\rho} f(t,\tau) dt\, d\tau \right| \right\}$$

and ρ stands for any measurable subset of $\bar{R}_i$. Then $f \in D(R_0)$ and for each subsegment $R \subset R_0$ we have

$$(D) \iint\limits_{R} f(t,\tau) dt\, d\tau = (D) \iint\limits_{R \cap E} f(t,\tau) dt\, d\tau + \sum_i \iint\limits_{R \cap \bar{R}_i} f(t,\tau) dt\, d\tau.$$

Proof. Let us consider the function

$$F(x,y) = \sum_i (D) \iint\limits_{\bar{R}_i \cap I(x,y)} f(t,\tau) dt\, d\tau,$$

where $I(x,y) = [a,x;c,y]$.

We shall show that F is continuous on R_0, i.e., F is continuous at any point (x_0, y_0) of R_0; for $\varepsilon > 0$ there must exist a $\delta(\varepsilon) > 0$ such that the inequality $|F(x_0+k,\, y_0+l) - F(x_0,y_0)| < \varepsilon$. must take place for all k and l, $|k| < \delta(\varepsilon)$, $|l| < \delta(\varepsilon)$. Then

$$\begin{aligned} \big|F(x_0+k,\, y_0+l) - F(x_0,y_0)\big| &= \Big| \sum_i (D) \iint_{\bar{R}_i \cap I(x_0+k, y_0+l)} f(t,\tau) dt\, d\tau - \\ &- \sum_i (D) \iint_{\bar{R}_i \cap I(x_0,y_0)} f(t,\tau) dt\, d\tau \Big| \le \sum_i \Big| (D) \iint_{\bar{R}_i \cap P_{x_0,y_0}} f(t,\tau) dt\, d\tau \Big|, \end{aligned}$$

where $P_{x_0,y_0} = [a, x_0; y_0, y_0+l] \cup [x_0, x_0+k; c, y_0+l]$.

To estimate the last sum let us consider two subsystems of intervals from the system $\{R_i\}_{i=\overline{1,\infty}}$: $I' = \{R_i'\}$ and $I'' = \{R_i''\}$ such that system I' satisfies the conditions:

$$\bar{R}_i' \cap P_{x_0,y_0} \neq \varnothing \text{ and } i > N$$

and system I'' the conditions

$$\bar{R}_i'' \cap P_{x_0,y_0} \neq \varnothing \text{ and } i \le N.$$

For the first system for all $\varepsilon > 0$ there is an $N_0 = N_0(\varepsilon)$ such that if $N > N_0$, then by the condition (1) we get the estimate

$$\sum_{i>N} \Big|(D) \iint\limits_{\bar{R}'_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\Big| < \sum_{i>N} O[D; f, \bar{R}'_i] < \frac{\varepsilon}{2}.$$

Now consider system I''. Since $i \leq N$, this system is finite. Let N^* be the number of members of this system. Then for $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|P_{x_0,y_0}| < \delta$ and $|P_{x_0,y_0} \cap \bar{R}''_i| < |P_{x_0,y_0}| < \delta$. Hence it follows

$$(D) \iint\limits_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau < \frac{\varepsilon}{2N^*}.$$

Now

$$\sum_{i \leq N} \Big|(D) \iint\limits_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\Big| \leq \frac{\varepsilon}{2N^*} \cdot N^* = \frac{\varepsilon}{2}.$$

Finally, we obtain

$$|F(x_0+k, y_0+l) - F(x_0,y_0)| \leq \sum_i \Big|(D) \iint_{\bar{R}_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\Big| =$$

$$= \sum_{i>N} \Big|(D) \iint_{\bar{R}'_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\Big| + \sum_{i \leq N} \Big|(D) \iint_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\Big| \leq \varepsilon.$$

Thus F is continuous on R_0.

Now for the fixed i let us consider any finite system of pairwise disjoint compact intervals of type 2 with respect to R_i, say, $\{S_k\}_{k=1}^n$. If $\sum_{k=1}^n |S_k| < \delta$, then, by the conditions of the lemma,

$$\sum_k |\Delta(F; S_k)| = \sum_k \Big| \iint\limits_{S_k} f(t,\tau)dt\,d\tau\Big| < \varepsilon.$$

Therefore $F \in ACG\ (\bar{R}_i)_{i=\overline{1,\infty}}$. Now we shall show that F is absolutely continuous on E (or, briefly, is AC, written as $F \in AC(E)$) (see Definition 3 or [8], p. 97). To this end we define the function G as follows:

$$G(x,y) = \begin{cases} 0, & \text{when} \quad (x,y) \in E, \\ \frac{1}{|\bar{R}_i|}(D) \iint_{\bar{R}_i} f(t,\tau)dt\,d\tau, & \text{when} \quad (x,y) \in R_i,\ i = \overline{1,\infty}. \end{cases}$$

We show that G is Lebesgue integrable on R_0.

$$\iint_{R_0} |G(x,y)| dx\,dy = \sum_i \iint_{\bar{R}_i} |G(x,y)| dx\,dy =$$
$$= \sum_i \iint_{R_i} \Big(\frac{1}{|\bar{R}_i|} (D) \Big| \iint_{\bar{R}_i} f(t,\tau) dt\,d\tau \Big| \Big) dx\,dy =$$
$$= \sum_i \Big| (D) \iint_{\bar{R}_i} f(t,\tau) dt\,d\tau \Big| \iint_{\bar{R}_i} \frac{dx\,dy}{|\bar{R}_i|} = \sum_i \Big| (D) \iint_{\bar{R}_i} f(t,\tau) dt\,d\tau \Big| < C.$$

Put

$$\Psi(x,y) = \int_a^x \int_c^y G(t,\tau) dt\,d\tau.$$

Since G is summable, for any $\varepsilon > a$ there is a $\delta > 0$ such that $|e| < \delta$ implies

$$\Big| \iint_e G(t,\tau) dt\,d\tau \Big| < \varepsilon \tag{2}$$

for each measurable subset $e \subset R_0$.

If we consider any finite system of pairwise disjoint compact intervals of type 2 with respect to E, say, $\{I_k\}_{k=1}^m$, and if $\sum_{k=1}^m |I_k| < \delta$, then

$$\sum_{k=1}^m \Big| \iint_{I_k} G(t,\tau) dt\,d\tau \Big| < \varepsilon.$$

We shall now show that Ψ is AC on E. Assume $I_k = [\alpha_k, \beta_k; \gamma_k, \delta_k]$. Then

$$\sum_{k=1}^m |\Delta\Psi(I_k)| = \sum_{k=1}^m \Big| \iint_{I_k} G(t,\tau) dt\,d\tau \Big| < \varepsilon.$$

It is not difficult to check that $F(x,y) = \Psi(x,y)$ when $(x,y) \in E$. Since F is ACG on each R_i and $F \in AC(E)$, it follows that F is ACG on R_0.

Moreover,
$DF(x,y) = D\Psi(x,y) = G(x,y) = 0$ almost everywhere on E
and
$DF(x,y) = f(x,y)$ almost everywhere on $R_0 \backslash E = \bigcup_i R_i$.

Thus F is a D-primitive of φ_1 defined as follows:

$$\varphi_1(x,y) = \begin{cases} f(x,y) & \text{when} \quad (x,y) \in \bigcup_i R_i, \\ 0 & \text{when} \quad (x,y) \in E. \end{cases}$$

On the other hand, we consider

$$\varphi_2(x,y)=\begin{cases}0 & \text{when} \quad (x,y)\in\bigcup_i R_i,\\ f(x,y) & \text{when} \quad (x,y)\in E.\end{cases}$$

Since $f\in D(E)$, we have $\varphi_2\in D(R_0)$ and $f\in D(R_0)$, $f(x,y)=\varphi_1(x,y)+\varphi_2(x,y)$.

Finally, we obtain the equality

$$\begin{aligned}(D)\iint_R f(t,\tau)dt\,d\tau &= (D)\iint_R \varphi_1(t,\tau)dt\,d\tau+(D)\iint_R \varphi_2(t,\tau)dt\,d\tau=\\ &=(D)\iint_{R\cap E}\varphi_1(t,\tau)dt\,d\tau+(D)\iint_{R\cap(\cup_i\bar R_i)}\varphi_1(t,\tau)dt\,d\tau+\\ &+(D)\iint_{R\cap E}\varphi_2(t,\tau)dt\,d\tau+(D)\iint_{R\cap(\cup_i\bar R_i)}\varphi_2(t,\tau)dt\,d\tau=\\ &=(D)\iint_{R\cap E} f(t,\tau)dt\,d\tau+(D)\iint_{R\cap(\cup_i\bar R_i)} f(t,\tau)dt\,d\tau=\\ &=(D)\iint_{R\cap E} f(t,\tau)dt\,d\tau+\sum_i(D)\iint_{R\cap\bar R_i} f(t,\tau)dt\,d\tau,\end{aligned}$$

where R is a subsegment of R_0. □

Lemma 2. *Let F_n be continuous on R_0, $n\in N$, and $\lim_{n\to\infty}F_n=F$, where F is also continuous. Then $\lim_{n\to\infty}O[F_n;R_0]\ge O(F;R_0)$.*

Proof. Since F is continuous, there exist points (x_1,y_1), (x_2,y_2) on R_0 such that $F(x_2,y_2)-F(x_1,y_1)=O(F;R_0)$. On the other hand, for any $\varepsilon>0$ there is an $N_0=N_0(\varepsilon)>0$ such that $F_n(x_i,y_i)-\varepsilon\le F(x_i,y_i)\le F_n(x_i,y_i)+\varepsilon$, $i=1,2$, for all $n>N_0$.

It follows that $F(x_2,y_2)-F(x_1,y_1)\le F_n(x_2,y_2)-F_n(x_1,y_1)+2\varepsilon$ and so $O(F;R_0)\le O(F_n;R_0)+2\varepsilon$ from which the result is immediate. □

Definition 3. Let $F_n:R_0\to R$ be continuous, $n\in N$. Then $\{F_n;\, n\in N\}$ is said to be a sequence of uniformly A_x-functions on the set $P\subset[a,b]$ if $P=\bigcup_k P_k$ and F_n is uniformly AC on each $H_k=P_k\times[c,d]$ (see [9]), i.e., for all $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that for any finite system of pairwise disjoint compact intervals of type 2 with respect to H_k, say, $\{R_i\}$, $i=\overline{1,m}$, the inequality $\sum_{i=1}^m|R_i|<\delta$ implies $\sum_{i=1}^m|\Delta(F_n;R_i)|<\varepsilon$ for all $n\in N$.

Lemma 3. *If $f_n\in D(R_0)$, $n\in N$, $E\subset[a,b]$ is a closed set and the sequence*

$$F_n(x,y)=\int\limits_a^x\int\limits_c^y f_n(t,\tau)dt\,d\tau,\quad n\in N,$$

is uniformly AC on $E' = E \times [\gamma, \delta]$, *then for all* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon) > 0$ *such that for all measurable sets,* e, $e \subset E$ *and* $|e| < \delta$, *we have*

$$\Big| \iint\limits_{e'} f_n(t,\tau)dt\,d\tau \Big| < \varepsilon, \quad n = \overline{1,\infty}, \quad \textit{where } e' = e \times [\gamma,\delta].$$

Proof. By the conditions of the lemma, if $\varepsilon > 0$, then there is a $\delta > 0$ such that for any finite system $\{R_k\}$, $k = \overline{1,m}$, of disjoint compact intervals of type 2 with respect to E' the inequality $\sum_{k=1}^m |R_k| < \delta$ implies

$$\sum_{k=1}^m |F_n(R_k)| = \sum_{k=1}^m \Big| \iint_{R_k} f_n(t,\tau)dt\,d\tau \Big| < \varepsilon \tag{3}$$

for all $n \in N$.

Let now e be a measurable subset of E with $|e| < \delta$. Then there are open sets G_m, $m \in N$, such that $G_m \supset G_{m+1} \supset e$, $m \in N$, and $\lim_{m\to\infty} |G_m| = |e|$. It can further be assumed that $G_m = \cup_{k=1}^\infty r_k^{(m)}$, $r_k^{(m)} \cap r_e^{(m)} = \varnothing$, $k \neq l$, for each m, where $r_k^{(m)}$ are open intervals. We set $e_m = E \cap G_m = \cup_k (E \cap r_k^{(m)})$. Denote by $\rho_{k,j}^{(m)}$, $j \in N$, contiguous intervals of $E \cap r_k^{(m)}$ on $r_k^{(m)}$. Then $G_m \backslash e_m = \cup_k \cup_j \rho_{k,j}^{(m)}$.

The endpoints of each $\rho_{k,j}^{(m)}$ lie on E. Moreover, for $\delta = \delta(\varepsilon) > 0$ there is an $N_0(\delta) = N_0 > 0$ such that if $m > N_0$, then $|G_m| < \delta$.

If $m > N_0$, from (3) we obtain

$$\begin{gathered} \sum_k \Big| \iint_{r_k^{(m)} \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau \Big| < \varepsilon, \\ \sum_k \sum_j \Big| \iint_{\rho_{k,j}^{(m)} \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau \Big| < \varepsilon \end{gathered} \tag{4}$$

for all $n \in N$.

Now, by Lemma 1, for $m > N_0$ we have

$$\iint_{r_k^{(m)} \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau =$$
$$= \iint_{(r_k^{(m)} \cap E) \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau + \sum_j \iint_{\rho_{k,j}^{(m)} \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau, \quad n = \overline{1,\infty}.$$

Performing the summation of the last expression over k, from (4) we obtain

$$\Big| \iint\limits_{e_m \times [\gamma,\delta]} f_n(t,\tau)dt\,d\tau \Big| < 2\varepsilon, \quad n = \overline{1,\infty},$$

and so

$$\Big|\iint\limits_{e'} f_n(t,\tau)dt\,d\tau\Big| < 2\varepsilon, \quad \overline{1,\infty}. \quad \blacksquare$$

Note that the following lemma of P. Romanovskii is very essential for our investigation.

Lemma 4. *Let $\mathfrak{F}$ be a nonempty system of open subintervals of the open interval $]a,b[$, having the following four properties:*

1) *If $]\alpha,\beta[\in\mathfrak{F}$ and $]\beta,\gamma[\in\mathfrak{F}$, then $]\alpha,\gamma[\in\mathfrak{F}$;*
2) *If $]\alpha,\beta[\in\mathfrak{F}$, then every subinterval of $]\alpha,\beta[$ also belongs to $\mathfrak{F}$;*
3) *If every proper open subinterval of $]\alpha,\beta[$ belongs to $\mathfrak{F}$, then $]\alpha,\beta[\in\mathfrak{F}$;*
4) *If all contiguous intervals on $[a,b]$ of a nonempty perfect set E from $]a,b[$ belong to $\mathfrak{F}$, then $\mathfrak{F}$ contains an interval $]\alpha,\beta[$ such that $]\alpha,\beta[\cap E\neq\varnothing$.*

Then $]a,b[\in\mathfrak{F}$.

Theorem. *If $f_n\in D(R_0)$, $\lim_{n\to\infty} f_n = f$ almost everywhere on R_0 and the sequence*

$$F_n(x,y)=\int\limits_a^x\int\limits_c^y f_n(t,\tau)dt\,d\tau, \quad n=\overline{1,\infty},$$

is uniformly continuous on R_0 and uniformly A_x on $[a,b]$, then $f\in D(R_0)$ and

$$\lim_{n\to\infty}\iint\limits_{R_0} f_n(t,\tau)dt\,d\tau=\iint\limits_{R_0} f(t,\tau)dt\,d\tau.$$

Proof. The subinterval $]\alpha,\beta[\subset]a,b[$ will be said to lie in the family of subintervals $\mathfrak{F}$ if for $]\alpha,\beta[$ and all all compact subintervals we have

$$\lim_{n\to\infty}\int\limits_\alpha^\beta\int\limits_\gamma^\delta f_n(t,\tau)dt\,d\tau=\int\limits_\alpha^\beta\int\limits_\gamma^\delta f(t,\tau)dt\,d\tau, \tag{*}$$

where $[\gamma,\delta]\subset[c,d]$.

The theorem will be proved if we show that $\mathfrak{F}$ satisfies the conditions of Romanovskii's lemma. The condition of the theorem implies that there exists a portion P such that $\{F_n\}_{n\geq 1}$ is uniformly AC on $P\times[c,d]$. Then due to Vitali's theorem on passage of limits through integrals and by virtue of some properties of the Denjoy integral there exists an interval $[\alpha^*,\beta^*,\gamma^*,\delta^*]$ for which the equation (*) is fulfilled. Thus $\mathfrak{F}\neq\varnothing$.

Now assume that $]\alpha,\beta[\in\mathfrak{F}$ and $]\beta,\eta[\in\mathfrak{F}$. Then $]\alpha,\eta[\in\mathfrak{F}$. Indeed,

$$\lim_{n\to\infty}\int\limits_{\alpha}^{\eta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau=\lim_{n\to\infty}\Big(\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau+\int\limits_{\beta}^{\eta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau\Big)=$$

$$=\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau+\int\limits_{\beta}^{\eta}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau=\int\limits_{\alpha}^{\eta}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau,$$

i.e., $]\alpha,\eta[\in\mathfrak{F}$.

Condition 2) follows from the definition of the family $\mathfrak{F}$.

Now let us consider condition 3). It is assumed that every open proper subinterval of $]\alpha,\beta[$ is contained in $\mathfrak{F}$. Then we must show that $]\alpha,\beta[\in\mathfrak{F}$.

Consider $]\lambda_m,\mu_m[\in\mathfrak{F}$, $]\lambda_m,\mu_m[\subset]\alpha,\beta[$, $\lambda_m\downarrow\alpha$, $\mu_m\uparrow\beta$, $m\to\infty$.

Since $\{F_n\}_{n\geq 1}$ is uniformly continuous on R_0, for all $\varepsilon>0$ there is an $N_0=N_0(\varepsilon)$ such that if $m>N_0$, then

$$\int\limits_{\lambda_m}^{\mu_m}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau-\varepsilon\leq\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau\leq\int\limits_{\lambda_m}^{\mu_m}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau+\varepsilon,$$

where $[\gamma,\delta]\subset[c,d]$ is arbitrary.

Since $]\lambda_m,\mu_m[\in\mathfrak{F}$, we have

$$\int\limits_{\lambda_m}^{\mu_m}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau-\varepsilon\leq\varliminf_{n\to\infty}\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau\leq$$

$$\leq\varlimsup_{n\to\infty}\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau\leq\int\limits_{\lambda_m}^{\mu_m}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau+\varepsilon. \tag{5}$$

Hence, $\varepsilon>0$ being arbitrary, (5) implies that

$$\lim_{n\to\infty}\int\limits_{\lambda_m}^{\mu_m}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau \tag{6}$$

exists and

$$\lim_{n\to\infty}\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f_n(t,\tau)dt\,d\tau=\int\limits_{\alpha}^{\beta}\int\limits_{\gamma}^{\delta} f(t,\tau)dt\,d\tau.$$

Thus $]\alpha,\beta[\in\mathfrak{F}$ and condition 3) is satisfied.

Finally, we shall check condition 4). It is assumed that $E \subset [a,b]$ is a closed set. Since the sequence $\{F_n\}_{n\geq 1}$ is uniformly A_x on $[a,b]$, there exists a portion $P = E \cap [\alpha,\beta]$ such that $\{F_n\}_{n\geq 1}$ is uniformly AC on $P \times [c,d]$.

Let $\{I_k =]\alpha_k, \beta_k[,\ k = \overline{1,\infty}\}$ be the contiguous intervals of P on $]\alpha,\beta[$ which are the members of $\mathfrak{F}$. By Lemma 1, where $I_k \times [\gamma,\delta]$ are the contiguous intervals of $P \times [\gamma,\delta]$, $k = \overline{1,\infty}$, we have

$$\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau = \iint_{P\times[\gamma,\delta]} f_n(t,\tau)dt\,d\tau + \sum_k \iint_{I_k\times[\gamma,\delta]} f_n(t,\tau)dt\,d\tau. \quad (7)$$

By Lemma 3 and Vitali's theorem on passage of limits through integrals

$$\lim_{n\to\infty} \iint_{P'} f_n(t,\tau)dt\,d\tau = \iint_{P'} f(t,\tau)dt\,d\tau, \quad (8)$$

where $P' = P \times [\gamma,\delta]$.

Since $\{F_n\}_{n\geq 1}$ is uniformly AC on P' for all $\varepsilon > 0$, there is an $N_0 = N_0(\varepsilon)$ such that if $k > N_0$, then

$$\sum_{k>N_0} O(F_n; I_k \times [\gamma,\delta]) < \varepsilon. \quad (9)$$

Since $I_k \in \mathfrak{F}$, (9) implies

$$\sum_{k>N_0} O(F; I_k \times [\gamma,\delta]) < \varepsilon, \sum_{k=1}^{\infty} O(F; I_k \times [\gamma,\delta]) < +\infty. \quad (10)$$

Thus $f \in D([\alpha,\beta;\gamma,\delta])$ and from (7), (8), (10) we obtain

$$\lim_{n\to\infty} \int_\alpha^\beta \int_\gamma^\delta f_n(t,\tau)dt\,d\tau = \int_\alpha^\beta \int_\gamma^\delta f(t,\tau)dt\,d\tau,$$

i.e., $]\alpha,\beta[\in \mathfrak{F}$ and $]\alpha,\beta[\cap E \neq \varnothing$. Therefore $\mathfrak{F}$ satisfies all conditions of Romanovskii's lemma and $]a,b[\in \mathfrak{F}$. $\square$

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