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ON SOME PROPERTIES OF SOLUTIONS OF SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The properties of solutions of the equation $u''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t))$ are investigated where $p_i : [a, +\infty[\to R$ $(i = 1, 2)$ are locally summable functions, $\tau_1 : [a, +\infty[\to R$ is a measurable function and $\tau_2 : [a, +\infty[\to R$ is a nondecreasing locally absolutely continuous one. Moreover, $\tau_i(t) \geq t$ $(i = 1, 2)$, $p_1(t) \geq 0$, $p_2^2(t) \leq (4 - \varepsilon)\tau_2'(t)p_1(t)$, $\varepsilon = const > 0$ and $\int_a^{+\infty}(\tau_1(t) - t)p_1(t)dt < +\infty$. In particular, it is proved that solutions whose derivatives are square integrable on $[a, +\infty[$ form a one-dimensional linear space and for any such solution to vanish at infinity it is necessary and sufficient that $\int_a^{+\infty} tp_1(t)dt = +\infty$.

Consider the differential equation

$$u''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t)), \tag{1}$$

where $p_i : [a, +\infty[\to R$ $(i = 1, 2)$ are locally summable functions, $\tau_i : [a, +\infty[\to R$ $(i = 1, 2)$ are measurable functions and

$$\tau_i(t) \geq t \quad \text{for} \quad t \geq a \quad (i = 1, 2). \tag{2}$$

We say that a solution u of the equation (1) is a *Kneser-type* solution if it satisfies the inequality $u'(t)u(t) \leq 0$ for $t \geq a_0$ for some $a_0 \in [a, +\infty[$. A set of such solutions is denoted by K. By W we denote a space of solutions of (1) that satisfy $\int_a^{+\infty} u'^2(t)dt < +\infty$. The results of [1, 2] imply that if $p_1(t) \geq 0$ for $t \geq a$ and the condition

$$(i) \quad \tau_i(t) \equiv t, \ \ (i = 1, 2), \quad \int_a^{+\infty} |p_2(t)|dt < +\infty,$$

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or

$$(ii) \quad p_2(t) \leq 0, \quad \text{for } t \geq 0, \quad \int_a^{+\infty} sp_1(s)ds < +\infty, \quad \int_a^{+\infty} \frac{s}{\tau_2(s)}|p_2(s)|ds < +\infty,$$

holds, then $W \supset K$ and K is a one-dimensional linear space. The case when the conditions (i) and (ii) are violated, the matter of dimension of K and W and their interconnection has actually remained unstudied. An attempt is made in this note to fill up this gap to a certain extent.

Theorem 1. *Let $\tau_i(t) \geq t$ $(i = 1, 2)$, $p_1(t) \geq 0$ for $t \geq a$,*

$$\int_a^{+\infty} [\tau_1(t) - t]p_1(t)dt < +\infty, \tag{3}$$

and let τ_2 be a nondecreasing locally absolutely continuous function satisfying

$$p_2^2(t) \leq (4 - \varepsilon)\tau_2'(t)p_1(t) \quad \text{for} \quad t \geq a\ , \tag{4}$$

where $\varepsilon = const > 0$. Then

$$W \subset K, \quad \dim W = 1. \tag{5}$$

Before proceeding to the proof of the theorem we shall give two auxiliary statements.

Lemma 1. *Let the conditions of Theorem 1 be fulfilled and let $a_0 \in [a, +\infty[$ be large enough for the equality*

$$\int_{a_0}^{+\infty} [\tau_1(s) - s]p_1(s)ds \leq 4\delta^2, \tag{6}$$

where $\delta = \frac{1}{4}[2 - (4 - \varepsilon)^{1/2}]$, to hold. Then any solution u of the equation (1) *satisfies*

$$\delta \int_t^x [u'^2(s) + p_1(s)u^2(s)]ds \leq u'(x)u(x) - u'(t)u(t) +$$

$$+(1 - \delta) \int_x^{\tau(x)} u'^2(s)ds \quad for \quad a_0 \leq t \leq x < +\infty\ , \tag{7}$$

where $\tau(x) = \text{ess sup}_{a_0 \le t \le x}[\max_{1\le i\le 2} \tau_i(x)]$. *Moreover, if* $u \in W$, *then*

$$u'(t)u(t) \le -\delta \int_t^{+\infty} [u'^2(s) + p_1(s)u^2(s)]ds \quad \textit{for} \quad t \ge a_0 \tag{8}$$

and

$$2\delta \int_t^{+\infty} (s-t)[u'^2(s) + p_1(s)u^2(s)]ds \le u^2(t) \quad \textit{for} \quad t \ge a_0. \tag{9}$$

Proof. Let u be any solution of the equation (1). Then

$$-u''(t)u(t) + p_1(t)u^2(t) = p_1(t)u(t) \int_{\tau_1(t)}^{t} u'(s)ds - p_2(t)u'(\tau_2(t))u(t).$$

Integrating this equality from t to x, we obtain

$$u'(t)u(t) - u'(x)u(x) + \int_t^x [u'^2(s) + p_1(s)u^2(s)]ds =$$

$$= \int_t^x \Big[p_1(s)u(s) \int_{\tau_1(s)}^{s} u'(y)dy\Big] ds - \int_t^x p_2(s)u'(\tau_2(s))u(s)ds.$$

However, in view of (4) and (6),

$$\int_t^x \Big[p_1(s)u(s) \int_{\tau_1(s)}^{s} u'(y)dy\Big] ds \le \delta \int_t^x p_1(s)u^2(s)ds +$$

$$+\frac{1}{4\delta}\Big[\int_t^x \big[\tau_1(s) - s\big]p_1(s)ds\Big]\Big[\int_t^{\tau(x)} u'^2(s)ds\Big] \le$$

$$\le \delta \int_t^x p_1(s)u^2(s)ds + \delta \int_t^{\tau(x)} u'^2(s)ds \quad \text{for} \quad a_0 \le t \le x < +\infty$$

and

$$-\int_t^x p_2(s)u'(\tau_2(s))u(s)ds \le$$

$$\leq 2(1-2\delta)\int_t^x \Big[p_1(s)u^2(s)\Big]^{1/2}\Big[\tau_2'(s)u'^2(\tau_2(s))\Big]^{1/2}ds \leq$$

$$\leq (1-2\delta)\int_t^x p_1(s)u^2(s)ds + (1-2\delta)\int_t^x \tau_2'(s)u'^2(\tau_2(s))ds \leq$$

$$\leq (1-2\delta)\int_t^x p_1(s)u^2(s)ds + (1-2\delta)\int_t^{\tau(x)} u'^2(s)ds$$

$$\text{for} \quad a_0 \leq t \leq x < +\infty.$$

Therefore

$$u'(t)u(t) - u'(x)u(x) + \int_t^x [u'^2(s) + p_1(s)u^2(s)]ds \leq$$

$$\leq (1-\delta)\int_t^x \big[u'^2(s)+p_1(s)u^2(s)\big]ds + (1-\delta)\int_x^{\tau(x)} u'^2(s)ds$$

$$\text{for} \quad a_0 \leq t \leq x < +\infty$$

and thus the inequality (7) holds.

Suppose now that $u \in W$. Then, as one can easily verify,

$$\liminf_{x\to+\infty} |u'(x)u(x)| = 0.$$

So (7) immediately implies (8). Integrating both sides of (8) from t to $+\infty$, we obtain the estimate (9). □

Lemma 2. *Let the conditions of Lemma* 1 *be fulfilled and there exist* $b \in]a_0, +\infty[$ *such that*

$$p_i(t) = 0 \quad \textit{for} \quad t \geq b \;\; (i = 1, 2). \tag{10}$$

Then for any $c \in R$ *there exists a unique solution of the equation* (1) *satisfying*

$$u(a_0) = c, \quad u'(t) = 0 \quad \textit{for} \quad t \geq b. \tag{11}$$

Proof. In view of (2) and (10), for any $\alpha \in R$ the equation (1) has a unique solution $v(\cdot;\alpha)$ satisfying $v(t;\alpha) = \alpha$ for $b \leq t < +\infty$. Moreover, $v(t;\alpha) = \alpha v(t;1)$. On the other hand, by Lemma 1 the function $v(\cdot;1) : [a_0, +\infty[\to R$ is non increasing and $v(a_0;1) \geq 1$. Therefore the function $u(\cdot) = \frac{c}{v(a_0;1)} v(a_0;\cdot)$ is a unique solution of (1), (11). □

Proof of Theorem 1. First of all we shall prove that for any $c \in R$ the equation (1) has at least one solution satisfying

$$u(a_0) = c, \quad \int\limits_{a_0}^{+\infty} u'^2(s)ds < +\infty. \tag{12}$$

For any natural k put

$$p_{ik}(t) = \begin{cases} p_i(t) & \text{for} \quad a_0 \le t \le a_0 + k \\ 0 & \text{for} \quad t > a_0 + k \end{cases} \quad (i = 1, 2). \tag{13}$$

According to Lemma 2, for any k the equation $u''(t) = p_{1k}(t)u(\tau_1(t)) + p_{2k}(t)u'(\tau_2(t))$ has a unique solution u_k satisfying

$$u_k(a_0) = c, \quad u'_k(t) = 0 \quad \text{for} \quad t \ge a + k. \tag{14}$$

On the other hand, by Lemma 1

$$|u_k(t)| \le |c| \ \text{for} \quad t \ge a_0, \quad 2\delta \int\limits_{a_0}^{+\infty} (s - a_0)u'^2_k(s)ds \le c^2. \tag{15}$$

Taking (2) and (13)–(15) into account, it is easy to show that the sequences $(u_k)_{k=1}^{+\infty}$ and $(u'_k)_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on each closed subinterval of $[a_0, +\infty[$. Therefore, by the Arzela-Ascoli lemma, we can choose a subsequence $(u_{k_m})_{m=1}^{+\infty}$ out of $(u_k)_{k=1}^{+\infty}$, which is uniformly convergent alongside with $(u'_{k_m})_{m=1}^{+\infty}$ on each closed subinterval of $[a, +\infty[$. By (13)–(15) the function $u(t) = \lim_{m\to+\infty} u_{k_m}(t)$ for $t \ge a$ is a solution of the problem (1), (12).

We have thus proved that $\dim W \ge 1$. On the other hand, by Lemma 1 any solution $u \in W$ satisfies (8) and is therefore a Kneser-type solution. To complete the proof it remains only to show that $\dim W \le 1$, i.e., that the problem (1), (12) has at most one solution for any $c \in R$. Let u_1 and u_2 be two artbitrary solutions of this problem and $u_0(t) = u_2(t) - u_1(t)$. Since $u_0 \in W$ and $u_0(a_0) = 0$, by Lemma 1

$$2 \int\limits_{a_0}^{+\infty} (s - a_0)u'^2_0(s)ds = 0 \ \text{and} \ u_0(t) = 0 \ \text{for} \ t \ge a_0,$$

i.e., $u_1(t) \equiv u_2(t)$. $\square$

Remark 1. The condition (4) of Theorem 1 cannot be replaced by the condition

$$p_2^2(t) \le (4 + \varepsilon)\tau'_2(t)p_1(t) \quad \text{for} \quad t \ge a. \tag{16}$$

Indeed, consider the equation

$$u''(t) = \frac{1}{(4+\varepsilon)t^2}u(t) - \frac{1}{t}u'(t), \tag{17}$$

satisfying all conditions of Theorem 1 except (4), instead of which the condition (16) is fulfilled. On the other hand, the equation (17) has the solutions $u_i(t) = t^{\lambda_i}$ $(i = 1, 2)$, where $\lambda_i = (-1)^i(4+\varepsilon)^{-\frac{1}{2}}$ $(i = 1, 2)$. Clearly, $u_i \in W$ $(i = 1, 2)$. Therefore in our case instead of (5) we have $K \subset W$, $\dim W = 2$.

Corollary 1. *Let the conditions of Theorem* 1 *be fulfilled. Let, moreover,*

$$p_2(t) \le 0 \quad for \quad t \ge a. \tag{18}$$

Then

$$K = W, \quad \dim K = 1. \tag{19}$$

Proof. Let $u \in K$. Then by virtue of (18) and the non-negativity of p_1 there exists $t_0 \in [a, +\infty[$ such that $u(t)u'(t) \le 0$, $u''(t)u(t) \ge 0$ for $t \ge t_0$. Hence

$$\int_{t_0}^{+\infty} u'^2(s)ds \le |u(t_0)u'(t_0)|.$$

Therefore $u \in W$. Thus we have proved that $W \supset K$. This fact, together with (5), implies (19). □

A solution u of the equation (1) will be called *vanishing at infinity* if

$$\lim_{t \to +\infty} u(t) = 0. \tag{20}$$

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. Then for any solution* $u \in W$ *to vanish at infinity it is necessary and sufficient that*

$$\int_a^{+\infty} sp_1(s)ds = +\infty. \tag{21}$$

Proof. Let $u \in W$. Then by Lemma 1 $u^2(t) \ge \eta$ for $t \ge a_0$, where $\eta = \lim_{t\to+\infty} u^2(t)$, and $\int_{a_0}^{+\infty}(s-a_0)p_1(s)u^2(s)ds \le u^2(a_0)/2\delta$. Hence it follows that (21) implies $\eta = 0$, i.e., u is a vanishing solution at infnity.

To complete the proof it is enough to establish that if

$$\int_a^{+\infty} sp_1(s)ds < +\infty, \tag{22}$$

then any nontrivial solution $u \in W$ tends to a nonzero limit as $t \to +\infty$. Let us assume the contrary: the equation (1) has a nontrivial solution $u \in W$ vanishing at infinity. Then by Lemma 1

$$u(t)u'(t) \le 0, \quad \rho(t) \le \eta^2 u^2(t) \quad \text{for} \quad t \ge a_0, \tag{23}$$

where

$$\rho(t) = \int_t^{+\infty} (s-t)\big[u'^2(s) + p_1(s)u^2(s)\big]ds, \quad \eta = (2\delta)^{-\frac{1}{2}}.$$

On the other hand, by (4), (20) and (22) we have

$$|u(t)| = \left| \int_t^{+\infty} (s-t)\big[p_1(s)u(\tau_1(s)) + p_2(s)u'(\tau_2(s))\big]ds \right| \le$$

$$\le \Big[\int_t^{+\infty} (s-t)p_1(s)ds \Big]^{1/2} \Big[\int_t^{+\infty} (s-t)p_1(s)u^2(\tau_1(s))ds \big]^{1/2} +$$

$$+2 \int_t^{+\infty} (s-t)[p_1(s)]^{1/2}[\tau_2'(s)]^{1/2}|u'(\tau_2(s)|ds \le$$

$$\le \Big[\int_t^{+\infty} (s-t)p_1(s) \Big]^{1/2} \Big[\int_t^{+\infty} (s-t)p_1(s)u^2(\tau_1(s))ds \Big]^{1/2} +$$

$$+2\Big[\int_t^{+\infty} (s-t)p_1(s)ds \Big]^{1/2} \Big[\int_t^{+\infty} (s-t)\tau_2'(s)u'^2(\tau_2(s))ds \Big]^{1/2}$$

$$\text{for} \quad t \ge a_0.$$

Hence by (2) and (23) we find

$$|u(t)| \le \Big[\int_t^{+\infty} (s-t)p_1(s)ds \Big|^{1/2} \Big[\int_t^{+\infty} (s-t)p_1(s)u^2(s)ds \Big]^{1/2} +$$

$$+2\Big[\int_t^{+\infty} (s-t)p_1(s)ds \Big]^{1/2} \Big[\int_t^{+\infty} (s-t)u'^2(s)ds \Big]^{1/2} \le$$

$$\le 3\eta \Big[\int_t^{+\infty} (s-t)p_1(s)ds \Big]^{1/2} |u(t)| \quad \text{for} \quad t \ge a_0$$

and therefore $u(t) = 0$ for $t \geq a_1$, where a_1 is a sufficiently large number. By virtue of (2) the last equality implies $u(t) = 0$ for $t \geq a$. But this is impossible, since by our assumption u is a nontrivial solution. The obtained contradiction proves the theorem. $\square$

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